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THE

FIRST THREE SECTIONS

OF

Newton's Principia;

WITH

COPIOUS NOTES AND ILLUSTRATIONS,

AND

A GREAT VARIETY OF DEDUCTIONS AND PROBLEMS.

Designed for the Use of Students.

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INTRODUCTION.

THE following Compilation was drawn up at a time when the difficulties, which usually present themselves on a first perusal of the Principia, were fresh in the recollection of its Author. Upon a late accidental revision of it he was induced to think that it might, if printed in a convenient form, prove an useful guide to those, who not enjoying the benefits of Academical or other instruction, are yet desirous of becoming acquainted with so much at least of the Principia, as is necessary to a clear comprehension of the more prominent and obvious laws of the Planetary System. Perhaps even to the regularly educated Student it may not be wholly unacceptable as a book of occasional reference; inasmuch as, besides the Commentary properly so called, it will be found to contain, carefully arranged under proper heads, all or most of those Problems and Deductions from the Text, which, after having been collected by the Student at the expence of much time and trouble, are usually entered without any great regard to order or connexion, in the pages of his Manuscript.

The following is the plan and arrangement of this Treatise.

I. Newton's text entire, with the exception of Props. 3, 5, and 17; Lemmas 12, 13, and 14, relating to well-known properties of the Conic Sections; a few of the Scholia; and the aliter proofs in the 2d and 3d Sections; all of which, as being of less general use and application, might, it was conceived, be omitted without injury to the work.

II. A general Introduction to the three Sections, comprising a concise account, with Examples, of the Methods of Exhaustions and Indivisibles, and the doctrine of Limits.

III. Notes explanatory of Newton's text. In this part, which forms the main body of the Treatise, the following method has been invariably adhered to. (*j*) Each Lemma and Proposition is prefaced, wherever the subject appeared to require it, with such introductory remarks as were thought necessary to prepare the reader for Newton's demonstration. (*jj*) The Lemma or Proposition itself, where any difficulty occurs, is explained in as distinct and familiar a way as the subject would admit of. (*jjj*) At the end of each will be found subjoined, under the appellation of Notes, such further remarks, deductions, and problems as the Proposition under consideration seemed naturally to suggest.

IV. A collection of Miscellaneous Problems, with their solutions.

The reader will observe that the short account given of the doctrine of Exhaustions and Indivisibles, and also Arts. 52, 53, and 54, on curvature, have been extracted almost wholly from Maclaurin; and as utility has been his sole object, the Compiler of the following sheets has throughout unreservedly borrowed from every valuable source within his reach.

Should this attempt be favourably received by those for whose use it is exclusively designed, and the Author's leisure permit, the 7th and 8th Sections may probably follow, upon precisely the same plan.

MATHEMATICAL PRINCIPLES

OF

Natural Philosophy.

SECTION I.

OF THE METHOD OF PRIME AND ULTIMATE RATIOS, BY THE HELP OF WHICH THE FOLLOWING PROPOSITIONS ARE DEMONSTRATED.

LEMMA I.

Quantities, and the ratios of quantities, which, in any finite time, tend continually to equality; and, before the end of that time, approach nearer to each other than by any given difference, become ultimately equal.

IF you deny it, let them be ultimately unequal; and let their ultimate difference be D. Therefore they cannot approach nearer to equality than by that given difference D. Which is against the supposition.

LEMMA II.

If in any figure A a c E, *terminated by the right lines* A a, A E, *and the curve* a c E, *there are inscribed any number of parallelograms* A b, B c, C d, *&c. contained under equal bases* A B, B C, C D, *&c., and the sides* B b, C c, D d, *&c. parallel to* A a, *the side of the figure; and the parallelograms* a K b l, b L c m, c M d n, *&c. are completed. Then, if the breadth of those parallelograms is diminished, and their number is augmented continually; I say, that the ultimate ratios, which the inscribed figure* A K b L c M d D, *the circumscribed figure* A a l b m c n d o E, *and the curvilinear figure* A a b c d E, *have to each other, are ratios of equality.*—(Fig. 1.)

For the difference of the inscribed and circumscribed figure is the sum of the parallelograms K *l*, L *m*, M *n*, D *o*, that is (because of the equality of all their bases,) the rectangle under one of their bases K *b*, and the sum of their altitudes A *a*; that is, the rectangle A B *l a*. But this rectangle, because its breadth A B is diminished indefinitely, becomes less than any given rectangle. Therefore (by Lem. I.) the inscribed and circumscribed, and much more the intermediate curvilinear figure become ultimately equal. Which was to be demonstrated.

LEMMA III.

The same ultimate ratios are also ratios of equality, when the breadths A B, B C, C D, *&c. of the parallelograms are unequal, and are all diminished indefinitely.*

For let A F be equal to the greatest breadth; and let the parallelogram F A *a f* be completed. This will be greater than the difference of the inscribed and circumscribed figures; but, because its breadth A F is diminished indefinitely, it will become less than any given rectangle. Which was to be demonstrated.

Cor. 1. Hence the ultimate sum of the evanescent parallelograms coincides in every part with the curvilinear figure.

Cor. 2. Much more does the rectilinear figure, which is comprehended under the chords of the evanescent arcs *a b*, *b c*, *c d*, &c. ultimately coincide with the curvilinear figure.

Cor. 3. As also the circumscribed rectilinear figure, which is comprehended under the tangents of the same arcs.

Cor. 4. And, therefore, these ultimate figures (as to their perimeters *a c* E,) are not rectilinear, but curvilinear limits of rectilinear figures.

LEMMA IV.

If in two figures A a c E, P p r T, *there are inscribed (as before) two series of parallelograms, an equal number in each; and, their breadths being diminished indefinitely, if the ultimate ratios of the parallelograms in one figure to those in the other, each to each respectively, are the same; I say, that those two figures* A a c E, P p r T, *are to each other in that same ratio.*—(Fig. 2.)

For, as the parallelograms in one are severally to the parallelograms in the other; so, by composition, is the sum of all in one to the sum of all in the other; and so is one figure to the other; because (by Lem. III.) the former figure is to the former sum, and the latter figure to the latter sum, in the ratio of equality. Which was to be demonstrated.

Cor. Hence, if two quantities of any kind are any how divided into an equal number of parts: and those parts, when their number is augmented, and their magnitude diminished indefinitely, have a given ratio to each other, the first to the first, the second to the second, and so on in order; the whole quantities will be, one to the other, in that same given ratio. For, if in the figures of this Lemma, the parallelograms are taken to each other in the ratio of the parts, the sum of the parts will always be, as the sum of the parallelograms; and, therefore, the number of the parallelograms and parts being augmented, and their magnitudes diminished indefinitely, those sums will be in the ultimate ratio of the parallelogram in one figure, to the correspondent parallelogram in the other; that is, (by the supposition) in the ultimate ratio of any part of the one quantity to the corresponding part of the other.

LEMMA V.

All homologous sides of similar figures, whether curvilinear or rectilinear, are proportional; and the areas are in the duplicate ratio of the homologous sides.

LEMMA VI.

If any arc A C B, *given in position, is subtended by its chord* A B, *and in any point* A, *in the middle of a continued curvature, is touch-*

ed by a right line AD, *produced both ways; then, if the points* A *and* B *approach one another and meet; I say that the angle* B A D, *contained between the chord and the tangent, will be diminished indefinitely, and will ultimately vanish.*—(Fig. 3.)

For, if that angle does not vanish, the arc A C B will contain with the tangent A D an angle equal to a rectilinear angle; and, therefore, the curvature at the point A will not be continued. Which is against the supposition.

LEMMA VII.

The same things being supposed, I say, that the ultimate ratio of the arc, the chord, and the tangent, to each other, is the ratio of equality.

For, while the point B approaches towards the point A, let A B and A D be considered as produced to the remote points *b* and *d*, and let *b d* be drawn parallel to the secant B D. Let the arc A *c b* be always similar to the arc A C B. Then, supposing the points A and B to coincide, the angle *d* A *b* will vanish, by the preceding Lemma; and, therefore, the right lines A *b*, A *d*, which are always finite, and the intermediate arc A *c b* will coincide, and become equal among themselves. Wherefore, the right lines AB, AD, and the intermediate arc ACB, which are always proportional to the former, will vanish; and will ultimately acquire the ratio of equality. Which was to be demonstrated.

Cor. 1.—(Fig. 4.) Whence, if through B be drawn BF parallel to the tangent, always cutting any right line AF, passing through A, in F; this line BF will ultimately have the ratio of equality to the evanescent arc ACB; because, completing the parallelogram AFBD, it always has the ratio of equality to AD.

Cor. 2. And, if through B and A more right lines are drawn, as BE, BD, AF, AG, cutting the tangent AD, and its parallel BF; the ultimate ratio of all the abscissas AD, AE, BF, BG, and of the chord, and arc AB, to each other, will be the ratio of equality.

Cor. 3. And, therefore, in all our reasonings about ultimate ratios, we may freely use any one of these lines for any other.

LEMMA VIII.

If the right lines AR, BR, *with the arc* ACB, *the chord* AB, *and the tangent* AD, *constitute three triangles* RAB, RACB, RAD, *and then the points* A *and* B *approach to each other; I say, that the ultimate form of* *the evanescent triangles is that of similitude, and the ultimate ratio that of equality.*— (Fig. 3.)

For, while the point B approaches towards the point A, consider always A B, A D, A R, as produced to the remote points *b*, *d*, and *r*; and *r b d*, as drawn parallel to R D; and let the arc A *c b* be always similar to the arc A C B. And, supposing the points A and B to coincide, the angle *b* A *d* will vanish; and, therefore, the three triangles *r* A *b*, *r* A *c b*, *r* A *d*, which are always finite, will coincide; and, on that account, become both similar and equal. Therefore the triangles R A B, R A C B, R A D, which are always similar and proportional to these, will ultimately become both similar and equal among themselves. Which was to be demonstrated.

Cor. And hence, in all our reasonings about ultimate ratios, we may indifferently use any one of these triangles for any other.

LEMMA IX.

If a right line A E, *and a curve line* A B C, *given in position, cut each other in a given angle* A; *and to that right line, in another given angle,* B D, C E *are ordinately applied, meeting the curve in* B, C; *and the* *points* B *and* C *together approach towards the point* A: *I say, that the areas of the triangles* A B D, A C E, *will ultimately be, one to the other, in the duplicate ratio of the sides.*—(Fig. 5.)

For, while the points B, C approach towards the point A, suppose always A D to be produced to the remote points *d* and *e*, so that A *d*, A *e*, may be proportional to A D, A E; and let the ordinates *d b*, *e c*, be erected parallel to the ordinates D B, E C, and meeting A B, A C produced in *b* and *c*. Let the curve A *b c* be drawn similar to the curve A B C; and also the right line A *g*, which may touch both curves in A, and cut the ordinates D B, E C, *d b*, *e c*, in F, G, *f*, *g*. Then, supposing the length A *e* to remain the same, let the points B and C meet in the point A; and, the angle *c* A *g* vanishing, the curvilinear areas A *b d*, A *c e*, will coincide with the rectilinear areas A *f d*, A *g e*; and, therefore, (by Lem. V.) will be in the duplicate ratio of the sides A *d*, A *e*. But the areas A B D, A C E, are always proportional to these areas; and the sides A D, A E to these sides. Therefore also, the areas A B D, A C E are ultimately in the duplicate ratio of the sides A D, A E. Which was to be demonstrated.

LEMMA X.

The spaces, which a body describes by any finite force urging it, whether that force is determined and immutable, or is continually augmented or continually diminished, are, in the very beginning of the motion, in the duplicate ratio of the times.

Let the times be represented by the lines A D, A E; and the velocities generated in those times by the ordinates D B, E C: and the spaces, described with these velocities, will be as the areas A B D, A C E, described by these ordinates; that is, at the very beginning of the motion (by Lem. IX.) in the duplicate ratio of the times A D, A E. Which was to be demonstrated.

Cor. 1. And hence it is easily inferred, that the errors of bodies, describing similar parts of similar figures in proportional times, which are generated by any equal forces, similarly applied to the bodies, and are measured by the distances of the bodies from those places of the similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times, are nearly in the duplicate ratio of the times in which they are generated.

Cor. 2. But the errors, which are generated by proportional forces, similarly applied, at similar parts of similar figures, are as the forces and the squares of the times jointly.

Cor. 3. The same thing is to be understood of any spaces whatsoever, described by bodies which are urged with different forces. These are, in the very beginning of the motion, as the forces and the squares of the times jointly.

Cor. 4. And, therefore, the forces are as the spaces described in the very beginning of the motion directly, and the squares of the times inversely.

Cor. 5. And the squares of the times are as the spaces described directly, and the forces inversely.

LEMMA XI.

The evanescent subtense of the angle of contact, in all curves, which at the point of contact have a finite curvature, is ultimately in the duplicate ratio of the subtense of the conterminous arc.—(Fig. 6.)

Case 1. Let A B be that arc, A D its tangent, B D the subtense of the angle of contact perpendicular to the tangent, A B the subtense of the arc. Let A G, B G be erected perpendicular to the subtense A B and the tangent A D, meeting in G; then let the points D, B, G, approach to the points *d*, *b*, *g*; and let I be the ultimate intersection of the lines B G, A G, supposing the points D, B, to approach continually to A. It is evident, that the distance G I may be less than any assignable. But, (from the nature of circles passing through the points A B G, A b g) $AB^2 = AG \times BD$, and $Ab^2 = Ag \times bd$; and therefore, the ratio of AB^2 to Ab^2 is compounded of the ratios of A G to A g, and of B D to b d. But, because G I may be assumed less than any assignable length, the ratio of A G to A g may differ from the ratio of equality, less than by any assignable difference; and, therefore, the ratio of AB^2 to Ab^2 may differ from the ratio of B D to b d, less than by any assignable difference. Therefore, by Lem. I. the ultimate ratio of AB^2 to Ab^2 is the same with the ultimate ratio of B D to b d. Which was to be demonstrated.

Case 2. Let B D be inclined to A D in any given angle, and the ultimate ratio of B D to b d will always be the same as before; and, therefore, the same as the ratio of AB^2 to Ab^2. Which was to be demonstrated.

Case 3. And, although the angle D is not given, but the right line B D converges to a given point, or is determined by any other condition whatever; yet the angles D, d, being determined by the same law, will always converge to equality, and approach nearer to each other than by any assigned difference; and by Lem. I. will be ultimately equal; and, therefore, the lines B D, b d are in the same ratio to each other as before. Which was to be demonstrated.

Cor. 1. Therefore, since the tangents A D, A d, the arcs A B, A *b*, and their sines B C, *b c*, become ultimately equal to the chords A B, A *b*; their squares also will ultimately be as the subtenses B D, *b d*.

Cor. 2. The same squares are also ultimately as the versed sines of the arcs, which bisect the chords, and converge to a given point. For those versed sines are as the subtenses B D, *b d*.

Cor. 3. And, therefore, the versed sine is in the duplicate ratio of the time, in which a body describes the arc with a given velocity.

Cor. 4. The rectilinear triangles A D B, A *d b* are ultimately in the triplicate ratio of the sides A D, A *d*; and in the sesquiplicate ratio of the sides D B, *d b*; as being in the compound ratio of the sides A D and D B, A *d* and *d b*. So also the triangles A B C, A *b c* are ultimately in the triplicate ratio of the sides B C, *b c*. What I call the sesquiplicate ratio is the subduplicate of the triplicate, which is compounded of the simple and subduplicate ratio.

Cor. 5. And, because D B, *d b*, are ultimately parallel, and in the duplicate ratio of A D, A *d*, the ultimate curvilinear areas A D B, A *d b* will be (by the nature of the parabola) two-thirds of the rectilinear triangles A D B, A *d b*; and the segments A B, A *b* will be one-third of the same triangles. And hence these areas, and these segments, will be in the triplicate ratio, as well of the tangents A D, A *d*, as of the chords and arcs A B, A *b*.

SCHOLIUM.

But, we have all along supposed the angle of contact to be neither indefinitely greater, nor indefinitely less, than the angles of contact, which circles contain with their tangents; that is, that the curvature at the point A is neither indefinitely small, nor indefinitely great; or, that the interval A I is of a finite magnitude. For D B may be taken as AD^3: in which case, no circle can be drawn through the point A, between the tangent A D, and the curve A B, and therefore the angle of contact will be indefinitely less than circular angles. And, by a like reasoning, if D B be made successively as AD^4, AD^5, AD^6, AD^7, &c. we shall have a series of angles of contact proceeding continually, whereof every succeeding series is indefinitely less than the preceding. And if D B be made successively as AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$, &c. we shall have another series of angles of contact, the first of which is of the same kind with those of circles, the second indefinitely greater, and every succeeding one indefinitely greater than the preceding. But, between any two of these angles, another series of intermediate angles of contact may be interposed, proceeding both ways indefinitely, whereof every succeeding angle shall be indefinitely greater, or indefinitely less than the preceding. As if, between the terms AD^2, and AD^3, there was interposed the series $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$,

&c. And again, between any two angles of this series, a new series of intermediate angles may be interposed, differing from one another by intervals indefinitely great. Nor is nature confined to any limit.

Those things, which have been demonstrated of curve lines, and the surfaces which they comprehend, are easily applied to the curve surfaces and contents of solids. But I premised these Lemmas to avoid the tediousness of deducing long demonstrations to an absurdity, according to the method of the ancient geometers. For demonstrations are rendered more concise by the method of indivisibles. But, because the hypothesis of indivisibles is somewhat harsh, and therefore that method is esteemed less geometrical, I chose rather to reduce the demonstrations of the following propositions to the prime and ultimate sums and ratios of nascent and evanescent quantities; that is, to the limits of those sums and ratios: and so to premise the demonstrations of those limits, as briefly as I could. For hereby the same thing is performed, as by the method of indivisibles; and those principles being demonstrated, we may now use them with more safety. Therefore, if hereafter I shall happen to consider quantities, as made up of particles, or shall use little curve lines for right ones, I would not be understood to mean indivisible, but evanescent divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios: and, that the force of such demonstrations always depends on the method laid down in the preceding Lemmas.

SECTION II.

OF THE INVENTION OF CENTRIPETAL FORCES.

PROPOSITION I.—THEOREM I.

That the areas, which revolving bodies describe by radii, drawn to an immoveable centre of force, do both lie in the same immoveable planes, and are proportional to the times in which they are described.—(Fig. 7.)

Let the time be divided into equal parts, and in the first part of time, let the body, by its power of persevering in its state of uniform motion in a right line, describe the right line A B. In the second part of time, the same would, if not hindered, proceed directly to *c*, describing the line B *c* equal to A B; so that by the radii A S, B S, *c* S, drawn to the centre, the equal areas A S B, B S *c*, would be described. But when the body is arrived at B, let a centripetal force act at once, with a strong impulse, and make the body turn aside from the right line

B *c*, and afterwards continue its motion along the right line B C. Draw *c* C parallel to B S, meeting B C in C; and, at the end of the second part of time, the body will be found in C, in the same plane with the triangle A S B. Join S C; and, because S B and C *c* are parallel, the triangle S B C will be equal to the triangle S B *c*, and therefore also to the triangle S A B. By a like argument, if the centripetal force acts successively in C, D, E, &c. making the body, in each single particle of time, to describe the several right lines C D, D E, E F, &c. they will lie in the same plane; and the triangle S C D will be equal to the triangle S B C, and S D E to S C D, and S E F to S D E. Therefore, in equal times, equal areas are described in one immoveable plane: and, by composition, any sums S A D S, S A F S, of those areas are to each other, as the times in which they are described. Let the number of those triangles be augmented, and their breadth diminished indefinitely; and (by Cor. 4. Lem. III.) their ultimate perimeter A D F will be a curve line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of the curve, will act continually; and any areas described S A D S, S A F S, which are always proportional to the times of description, will, in this case also, be proportional to those times. Which was to be demonstrated.

Cor. 1. The velocity of a body, attracted towards an immoveable centre in spaces void of resistance, is reciprocally as the perpendicular let fall from that centre on the right line that touches the orbit. For the velocities in those places A, B, C, D, E, are as the bases AB, BC, CD, DE, EF, of equal triangles; and these bases are reciprocally as the perpendiculars let fall upon them.

Cor. 2. If the chords A B, B C, of two arcs, successively described in equal times by the same body in spaces void of resistance, are completed into a parallelogram A B C V, and the diagonal B V of this parallelogram, in the position which it ultimately acquires, when those arcs are diminished indefinitely, is produced both ways, it will pass through the centre of force.

Cor. 3. If the chords A B, B C, and D E, E F, of arcs, described in equal times in spaces void of resistance, are completed into the parallelograms A B C V, D E F Z; the forces in B and E are to each other in the ultimate ratio of the diagonals B V and E Z, when those arcs are diminished indefinitely. For the motions B C, and E F of the body are compounded of the motions B *c*, B V, and E *f*, E Z: but B V and E Z, equal to C *c* and F *f*, in the demonstration of this proposition, were generated by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

Cor. 4. The forces, by which bodies in spaces void of resistance are drawn back from their rectilinear motions, and turned into curvilinear orbits, are to each other, as those versed sines of arcs described in equal times, which converge to the centre of force, and bisect the chords, when those arcs are diminished indefinitely. For such versed sines are half the diagonals mentioned in Cor. 3.

Cor. 5. And, therefore, those forces are to the force of gravity, as the said versed sines, to the versed sines perpendicular to the horizon of the parabolic arcs, which projectiles describe in the same time.

Cor. 6. The same things hold good when the planes in which the bodies are moved, together with the centres of force, which are placed in those planes, are not at rest, but move uniformly in a right line.

PROPOSITION II.—THEOREM II.

Every body that moves in any curve line described in a plane, and by a radius drawn to a poiut, either immoveable, or moving forward with an uniform rectilinear motion, describes about that point areas proportional to the times, is urged by a centripetal force tending to that point.

Case 1. For every body, that moves in a curve line, is turned aside from its rectilinear course by the action of some force that impels it. And that force by which the body is turned off from its rectilinear course, and is made to describe, in equal times, the very small equal triangles S A B, S B C, S C D, &c.

about the immoveable point S, acts, in the place B, in the direction of a line parallel to *c* C; that is, in the direction of the line B S; and in the place C, in the direction of a line parallel to *d* D, that is, in the direction of the line C S, &c. It acts, therefore, always in the direction of lines tending to that immoveable point S. Which was to be demonstrated.

Case 2. And it is indifferent, whether the surface in which a body describes a curvilinear figure is quiescent; or moves, together with the body, with the figure described, and its point S, uniformly in a right line.

Cor. 1. In spaces or mediums void of resistance, if the areas are not proportional to the times, the forces do not tend to the point in which the radii meet; but deviate therefrom *in consequentia*, or towards the part to which the motion is directed, if the description of areas is accelerated; but *in antecedentia*, if retarded.

Cor. 2. And, even in resisting mediums, if the description of areas is accelerated, the directions of the forces deviate from the concourse of the radii, towards the part to which the motion tends.

SCHOLIUM.

A body may be urged by a centripetal force compounded of several forces. In this case, the meaning of the proposition is, that the force, which is compounded of all, tends to the point S. But, if any force acts perpetually in the direction of lines perpendicular to the described surface, this force will make the body to deviate from the plane of its motion: but it will neither augment nor diminish the quantity of the described surface, and is therefore to be neglected in the composition of forces.

PROPOSITION IV.—THEOREM IV.

That the centripetal forces of bodies, which by an equable motion describe different circles, tend to the centres of the same circles; and are to each other, as the squares of the arcs described in equal times, applied to the radii of the circles.

These forces tend to the centres of the circles, (Prop II. and Cor. 2. Prop. I.) and are to each other as the versed sines of arcs, described in equal times indefinitely small (by Cor. 4. Prop. I.); that is, as the squares of the same arcs, applied to the diameters of the circles, (by Lem. VII.) and, therefore, since these arcs are as the arcs described in any equal times, and the diameters are as the radii; the forces will be as the squares of any arcs described in the same time, applied to the radii of the circles. Which was to be demonstrated.

Cor. 1. Since those arcs are as the velocities of the bodies, the centripetal forces are in a ratio compounded of the duplicate ratio of the velocities directly, and of the simple ratio of the radii inversely.

Cor. 2. And, since the periodical times are in a ratio compounded of the ratio of the radii directly, and the ratio of the velocities inversely; the centripetal forces are in a ratio compounded of the ratio of the radii directly, and the duplicate ratio of the periodical times inversely.

Cor. 3. Whence it appears, that if the periodical times are equal, and therefore the velocities are as the radii; the centripetal forces will be also as the radii; and the contrary.

Cor. 4. If the periodical times and the velocities are both in the subduplicate ratio of the radii; the centripetal forces will be equal among themselves: and the contrary.

Cor. 5. If the periodical times are as the radii, and therefore the velocities equal; the centripetal forces will be reciprocally as the radii: and the contrary.

Cor. 6. If the periodical times are in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally in the subduplicate ratio of the radii; the centripetal forces will be inversely in the duplicate ratio of the radii: and the contrary.

Cor. 7. And universally, if the periodical time is as any power R^n of the radius R, and therefore the velocity reciprocally as the power R^{n-1} of the radius; the centripetal force will be reciprocally as the power of the radius R^{2n-1}: and the contrary.

Cor. 8. The same things all follow concerning the times, the velocities, and forces, by which bodies describe the similar parts of any similar figures, that have their centres in a similar position within these figures, by applying the demonstration of the preceding cases to those. And the application is made, by substituting the equable description of areas for equable motion, and using the distances of the bodies from the centres for the radii.

Cor. 9. From the same demonstration it likewise follows, that the arc, which a body, uniformly revolving in a circle with a given centripetal force, describes in any time, is a mean proportional between the diameter of the circle, and the space, which the same body, descending by the same given force, would describe in the same given time.

SCHOLIUM.

The case of the sixth corollary is applicable to the celestial bodies (as our countrymen Sir Christopher Wren, Dr. Hooke, and Dr. Halley, have severally observed); and, therefore, in what follows, I intend to treat more at large of those things which relate to a centripetal force decreasing in a duplicate ratio of the distances from the centres.

Moreover, by means of the preceding proposition and its corollaries, we may discover the proportion of a centripetal force to any other known force, such as that of gravity. For if a body, by means of its gravity, revolves in a circle concentric to the earth, this gravity is its centripetal force. But, from the descent of heavy bodies, the time of one entire revolution, as well as the arc described in any given time, is given (by Cor. 9 of this Prop.) And by such propositions, Mr. Huygens, in his excellent book *De Horologio Oscillatorio*, has compared the force of gravity with the centrifugal forces of revolving bodies.

PROPOSITION VI.—THEOREM V.

If a body, in a space void of resistance, revolves in any orbit about an immoveable centre, and in an indefinitely small time describes any nascent arc; and the versed sine of that arc is supposed to be drawn, which may bisect the chord, and being produced may pass through the centre of force; the centripetal force, in the middle of the arc, will be as the versed sine directly, and the square of the time inversely.

For the versed sine, in a given time, is as the force (by Cor. 4. Prop. I.) and increasing the time in any ratio, because the arc will be increased in the same ratio, the versed sine will be increased in the duplicate of that ratio, (by Cor. 2 and 3, Lem. XI.); and therefore is as the force, and the square of the time.

Subduct on both sides the duplicate ratio of the time, and the force will be as the versed sine directly, and the square of the time inversely. Which was to be demonstrated.

And the same thing is also easily demonstrated by Cor. 4. Lem. X.

Cor. 1.—(Fig. 8.) If a body P, revolving about the centre S, describes a curve line A P Q, and a right line Z P R touches that curve in any point P; and, from any other point Q of the curve, Q R is drawn parallel to the distance S P, meeting the tangent in R; and Q T is drawn perpendicular to the distance S P; the centripetal force will be reciprocally as the solid $\frac{SP^2 \times QT^2}{QR}$; if the solid is taken of that magnitude which it ultimately acquires, supposing the points P and Q continually to approach to each other. For Q R is equal to the versed sine of double the arc Q P, in whose middle is P: and double the triangle S Q P, or S P × Q T is proportional to the time, in which that double arc is described; and therefore may be used for the exponent of the time.

Cor. 2. By a like reasoning the centripetal force is reciprocally as the solid $\frac{SY^2 \times QP^2}{QR}$; if S Y is a perpendicular, let fall from the centre of force on P R, the tangent of the orbit. For the rectangles S Y × Q P and S P × Q T are equal.

Cor. 3. If the orbit is either a circle, or touches or cuts a circle concentrically, that is, contains with a circle an indefinitely small angle of contact or section; having the same curvature and the same radius of curvature at the point P; and if P V is a chord of this circle, drawn from the body through the centre of forces; the centripetal force will be reciprocally as the solid $SY^2 \times PV$. For P V is $\frac{QP^2}{QR}$

Cor. 4. The same things being supposed, the centripetal force is as the square of the velocity directly, and that chord inversely. For the velocity is reciprocally as the perpendicular S Y, by Cor. 1, Prop. I.

Cor. 5. Hence, if any curvilinear figure A P Q is given; and therein a point S is also given, to which a centripetal force is perpetually directed; the law of centripetal force may be found, by which the body P, continually drawn back from a rectilinear course, will be retained in the perimeter of that figure, and will describe the same by a perpetual revolution. That is, we are to find by computation, either the solid $\frac{SP^2 \times QT^2}{QR}$, or the solid $SY^2 \times PV$, reciprocally proportional to this force. Examples of this we shall give in the following Problems.

PROPOSITION VII.—PROBLEM II.

Let a body revolve in the circumference of a *circle; it is required to find the law of centripetal force tending to any given point.*—(Fig. 9.)

Let V Q P A be the circumference of the circle; S the given point, to which the force tends, as to a centre; P the body moving in the circumference; Q the next place into which it is to move, and P R Z the tangent of the circle at the preceding place. Through the point S let the chord P V be drawn; and, the diameter V A of the circle being drawn, let A P be joined; and let fall Q T perpendicular to S P, which produced may meet the tangent P R in Z; and lastly, through the point Q let L R be drawn, which may be parallel to S P, and may both meet the circle in L, and the tangent P Z in R. And, because of the similar triangles Z Q R, Z T P, V P A, RP^2, that is QRL will be to QT^2, as AV^2 to PV^2. And, therefore, $\frac{QRL \times PV^2}{AV^2}$, is equal to QT^2. Let these equals be multiplied into $\frac{SP^2}{QR}$, and the points P and Q continually approaching, for R L write P V. Thus we shall find $\frac{SP^2 \times PV^3}{AV^2} = \frac{SP^2 \times QT^2}{QR}$. Therefore (by Cor. 1 and 5, Proposition VI.) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is (because AV^2 is given) reciprocally as the square of the distance or altitude S P, and the cube of the chord P V jointly. Which was to be found.

Cor. 1. Hence, if the given point S, to which the centripetal force always tends, is placed in the circumference of this circle, suppose at V, the centripetal force will be reciprocally as the quadrato-cube (or fifth power) of the altitude S P.

Cor. 2.—(Fig. 10.) The force by which the body P in the circle A P T V revolves about the centre of force S, is to the force by which the same body P may revolve in the same circle, and in the same periodical time, about any other centre of force R, as $RP^2 \times SP$, to the cube of the right line S G, which is drawn from the first centre of force S, to the tangent of the orbit P G, and is parallel to the distance P R of the body from the second centre of force R.

For, by the construction of this proposition, the former force is to the latter, as $RP^2 \times PT^3$ to $SP^2 \times PV^3$; that is, as $SP \times RP^2$ to $\frac{SP^3 \times PV^3}{PT^3}$; or (because of the similar triangles P S G, T P V) to SG^3.

Cor. 3. The force, by which the body P in any orbit revolves about the centre of force S, is to the force, by which the same body P may revolve in the same orbit, and in the same periodical time, about any other centre of force R, as the solid $SP \times RP^2$, contained under the distance of the body from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line S G, which is drawn from the first centre of force S to the tangent P G of the orbit, and is parallel to the distance R P of the body from the second centre of force R. For the forces in this orbit, at any point P, are the same as in a circle of the same curvature.

PROPOSITION VIII.—PROBLEM III.

Let a body move in the semi-circumference P Q A ; *it is required to find the law of centripetal force, tending to a point* S, *so remote, that all lines* P S, R S *drawn thereto, may be taken for parallel.*—(Fig. 11.)

From C, the centre of the semi-circle, let the semi-diameter C A be drawn, cutting those parallels perpendicularly in M and N, and let C P be joined. Because of the similar triangles C P M, P Z T, and R Z Q, CP^2 is to PM^2, as PR^2 to QT^2; and, from the nature of the circle, PR^2 is equal to the rectangle $QR \times \overline{RN + QN}$; or, the points P and Q continually approaching, to the rectangle $QR \times 2PM$. Therefore CP^2 is to PM^2, as $QR \times 2PM$ to QT^2; therefore $\frac{QT^2}{QR} = \frac{2PM^3}{CP^2}$, and $\frac{QT^2 \times SP^2}{QR}$ $= \frac{2PM^3 \times SP^2}{CP^2}$. And therefore (by Cor. 1 and

5, Prop. VI.) the centripetal force is reciprocally as $\frac{2\,PM^3 \times SP^2}{CP^2}$; that is, (neglecting the given ratio $\frac{2\,SP^2}{CP^2}$) reciprocally as PM^3. Which was to be found.

The same thing is likewise easily collected from the preceding proposition.

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And, by a like reasoning, a body will be found to move in an ellipse, or even in an hyperbola, or parabola, by a centripetal force, which is reciprocally as the cube of the ordinate, directed to a centre of force, at a very great distance.

PROPOSITION IX.—PROBLEM IV.

Let a body revolve in a spiral P Q S *cutting all the radii* S P, S Q, *&c. in a given angle, it is required to find the law of centripetal force, tending to the centre of that spiral.*— (Fig. 12.)

Let the indefinitely small angle P S Q be given; and because all the angles are given, the species of the figure of S P R Q T will be given. Therefore the ratio $\frac{QT}{QR}$ is given; and $\frac{QT^2}{QR}$ is as Q T; that is, (because the species of that figure is given,) as S P. But if the angle P S Q is any way changed, the right line Q R, subtending the angle of contact Q P R (Lem. XI.) will be changed in the duplicate ratio of P R or Q T. Therefore the ratio $\frac{QT^2}{QR}$ remains the same as before; that is, as S P. Therefore $\frac{QT^2 \times SP^2}{QR}$ is as SP^3, and (by Cor. 1. and 5, Prop. VI.) the centripetal force is reciprocally as the cube of the distance S P. Which was to be found.

PROPOSITION X.—PROBLEM V.

Let a body revolve in an ellipsis: it is required to find the law of centripetal force, tending to the centre of the ellipsis.—(Fig. 13.)

Let C A, C B be semi-axes of the ellipsis, G P, D K other conjugate diameters; P F, Q T perpendiculars to those diameters; Q v an ordinate to the diameter G P; and if the parallelogram Q v P R is completed, the rectangle P v G will be to Qv^2, as PC^2 to CD^2; and (because of the similar triangles Q v T, P C F) Qv^2 is to QT^2, as PC^2 to PF^2; and by composition, the ratio of $P v G$ to $Q T^2$ is compounded of the ratio of $P C^2$ to $C D^2$, and of the ratio of $P C^2$, to $P F^2$; that is, $v G$ is to $\frac{Q T^2}{P v}$ as $P C^2$ to $\frac{C D^2 \times P F^2}{P C^2}$. Substitute $Q R$ for $P v$, and (by Lem. XII.) $B C \times C A$ for $C D \times P F$, also (the points P and Q continually approaching) $2 P C$ for $v G$; and multiplying the extremes and means together, we shall have $\frac{Q T^2 \times P C^2}{Q R}$ equal to $\frac{2 B C^2 \times C A^2}{P C}$. Therefore (by Cor. 5, Prop. VI.) the centripetal force is reciprocally as $\frac{2 B C^2 \times C A^2}{P C}$; that is (because $2 B C^2 \times C A^2$ is given) reciprocally as $\frac{1}{P C}$; that is, directly as the distance P C..... Which was to be found.

Cor. 1. And therefore, the force is as the distance of the body from the centre of the ellipse; and, on the contrary, if the force is as the distance, the body will move in an ellipse, whose centre coincides with the centre of force; or perhaps in a circle, into which the ellipse may be changed.

Cor. 2. And the periodical times of the revolutions made in all ellipses whatsoever about the same centre will be equal. For those times in similar ellipses are equal (by Cor. 3. and 8. Prop. IV.) but, in ellipses that have their greater axis common, they are to each other, as the whole areas of the ellipses directly, and the parts of the areas described in the same time inversely; that is, as the less axes directly, and the velocities of the bodies in the principal vertices inversely; that is, as those less axes directly, and the ordinates to the same point of the common axis inversely; and therefore (because of the equality of the direct and inverse ratios) in the ratio of equality.

SECTION III.

OF THE MOTION OF BODIES IN ECCENTRIC OR CONIC STATIONS.

PROPOSITION XI.—PROBLEM VI.

Let a body revolve in an ellipsis: it is required to find the law of centripetal force tending to the focus of the ellipsis.—(Fig. 14.)

LET S be the focus of the ellipsis. Draw S P, cutting the diameter D K of the ellipsis in E, and the ordinate Q v in x; and let the parallelogram Q x P R be completed. It is evident that E P is equal to the greater semi-axis A C: for, drawing H I from the other focus H of the ellipsis, parallel to E C, because C S, C H are equal, E S, E I will be also equal; so that E P is half the sum of P S, P I, that is, (because of the parallels H I, P R, and the equal angles I P R, H P Z,) of P S, P H; which taken together are equal to the whole axis 2 A C. Let Q T be perpendicular to S P, and putting L for the principal *latus* *rectum* of the ellipsis (or for $\frac{2\,BC^2}{AC}$,) $L \times QR$ will be to $L \times Pv$, as QR to Pv; that is, as PE, or AC to PC; and $L \times Pv$, to GvP, as L to Gv; and GvP to Qv^2 as PC^2 to CD^2; and (by Cor 2. Lem. VII.) the points Q and P continually approaching without end, Qv^2 is to Qx^2 in the ratio of equality; and Qx^2, or Qv^2, is to QT^2 as EP^2 to PF^2; that is, as CA^2 to PF^2; or, (by Conics) as CD^2 to CB^2. And compounding all these ratios together, $L \times QR$ is to QT^2, as $AC \times L \times PC^2 \times CD$, or $2\,CB^2 \times PC^2 \times CD^2$, to $PC \times Gv \times CD^2 \times CB^2$, or as $2\,PC$ to Gv. But, the points Q and P continually approaching without end, $2\,PC$ and Gv are equal. Therefore the quantities $L \times QR$ and QT^2 proportional to these, are also equal. Let these equals be multiplied into $\frac{SP^2}{QR}$, and $L \times SP^2$ will become equal to $\frac{SP^2 \times QT^2}{QR}$. Therefore (by Cor. 1. and 5. Prop. VI.) the centripetal force is reciprocally as $L \times SP^2$; that is, reciprocally in the duplicate ratio of the distance SP. Which was to be found.

PROPOSITION XII.—PROBLEM VII.

Let a body move in an hyperbola: it is requi-

red to find the law of centripetal force tending to the focus of that figure.—(Fig. 15.)

Let C A, C B be the semi-axes of the hyperbola; P G, K D other conjugate diameters; P F a perpendicular to the diameter K D; and Q v an ordinate to the diameter G P. Let S P be drawn cutting the diameter D K in E, and the ordinate Q v in x, and let the parallelogram Q R P x be completed. It is evident, that E P is equal to the semi-transverse axis A C; for, drawing H I from the other focus H of the hyperbola, parallel to E C, because C S, C H are equal, E S, E I will be also equal; so that E P is half the difference of P S, P I; that is (because of the parallels I H, P R, and the equal angles I P R, H P Z) of P S, P H; the difference of which is equal to the whole axis 2 A C. Let Q T be perpendicnlar to S P. And the principal *latus rectum* of the hyperbola (that is $\frac{2\,BC^2}{AC}$,) being called L, we shall have L × Q R to L × P v, as Q R to P v, or P x to P v; that is (because of the similar triangles P x v, P E C), as P E to P C, or A C to P C. Also L × P v will be to G v × P v, as L to G v; and (by the properties of the conic sections) the rectangle G v P is to Q v^2, as P C^2 to C D^2; and (by Cor. 2, Lem. VII.) Q v^2 to Q x^2, the points Q and P continually approaching without end, becomes a ratio of equality; and Q x^2 or Q v^2 is to Q T^2, as E P^2 to P F^2; that is, as C A^2 to P F2, or (by Conics) as C D^2 to

CB^2: and, compounding all these ratios together, $L \times QR$ is to QT^2, as $AC \times L \times PC^2 \times CD^2$, or $2\,CB^2 \times PC^2 \times CD^2$ to $PC \times Gv \times CD^2 \times CB^2$; or as $2\,PC$ to Gv. But the points P and Q continually approaching without limit, $2\,PC$ and Gv are equal. Therefore the quantities $L \times QR$ and QT^2, proportional to them, are also equal. Let these equals be multiplied into $\frac{SP^2}{QR}$, and $L \times SP^2$ will be equal to $\frac{SP^2 \times QT^2}{QR}$. Therefore, (by Cor. 1 and 5, Prop. VI.) the centripetal force is reciprocally as $L \times SP^2$; that is, reciprocally in the duplicate ratio of the distance S P. Which was to be found.

PROPOSITION XIII.—PROBLEM VIII.

Let a body move in the perimeter of a parabola: it is required to find the law of centripetal force, tending to the focus of that figure. —(Fig. 16.)

Let P be the body in the perimeter of the parabola, and from the place Q, into which the body is moving, draw Q R parallel, and Q T perpendicular to S P; as also Q v parallel to the tangent, and meeting both the diameter P G in v, and the distance S P in x.

Now, because of the similar triangles $P\,x\,v$, $S\,P\,M$, and the equal sides $S\,P$, $S\,M$ of the one, the sides $P\,x$ or $Q\,R$ and $P\,v$ of the other are also equal. But (by the properties of the conic sections) the square of the ordinate $Q\,v$ is equal to the rectangle under the *latus rectum*, and the segment $P\,v$ of the diameter; that is, (by Conics) to the rectangle $4\,P\,S \times P\,v$, or $4\,P\,S \times Q\,R$; and, the points P and Q approaching without limit, the ratio of $Q\,v$ to $Q\,x$ (by Cor. 2. Lem. VII.) becomes the ratio of equality. Therefore $Q\,x^2$, in this case, becomes equal to the rectangle $4\,P\,S \times Q\,R$. But (because of the similar triangles $Q\,x\,T$, $S\,P\,N$) $Q\,x^2$ is to $Q\,T^2$, as $P\,S^2$ to $S\,N^2$; that is, (by Conics) as $P\,S$ to $S\,A$; that is, as $4\,P\,S \times Q\,R$ to $4\,S\,A \times Q\,R$, and therefore (by Prop. IX. Lib. V. Elem.) $Q\,T^2$, and $4\,S\,A \times Q\,R$ are equal. Multiply these equals into $\frac{S\,P^2}{Q\,R}$, and $\frac{S\,P^2 \times Q\,T^2}{Q\,R}$ will become equal to $SP^2 \times 4\,S\,A$: and therefore (by Cor 1. and 5. Prop. VI.) the centripetal force is reciprocally as $S\,P^2 \times 4\,S\,A$; that is, because $4\,S\,A$ is given, reciprocally in the duplicate ratio of the distance S P. Which was to be found.

Cor. 1. From the three last propositions it follows, that if any body P goes from a place P, with any velocity, in the direction of any right line P R, and at the same time is urged by the action of a centripetal force, which is reciprocally proportional to the square of the distance of the places from the centre; this body will move in one of the conic sections, having its focus in the centre of force; and the contrary. For, the focus, the point of contact, and the position of the tangent being given, a conic section may be described, which at that point shall have a given curvature. But the curvature is given from the centripetal force and the velocity of the body being given, and two orbits, mutually touching each other, cannot be described by the same centripetal force, and the same velocity.

Cor. 2. If the velocity, with which the body goes from its place P, is such, that in any indefinitely small moment of time the line P R may be thereby described; and the centripetal force is such, as in the same time to move that body through the space Q R; the body will move in one of the conic sections; whose principal *latus rectum* is the limit, to which the quantity $\frac{QT^2}{QR}$ approaches, while the lines P R, Q R are continually diminished.

In these corollaries I consider the circle as an ellipsis; and I except the case, where the body descends to the centre in a right line.

PROPOSITION XIV.—THEOREM VI.

If several bodies revolve about one common centre, and the centripetal force is recipro-

cally in the duplicate ratio of the distance of places from the centre; I say, that the principal latera recta *of their orbits are in the duplicate ratio of the areas, which the bodies, by radii drawn to the centre, describe in the same time.*—(Fig. 8.)

For (by Cor. 2, Prop. XIII.) the *latus rectum* L is equal to the limit, to which the quantity $\frac{QT^2}{QR}$ approaches, while the distance of P and Q is continually diminished. But the small line Q R, in a given time, is as the generating centripetal force; that is, (by supposition) reciprocally as SP^2. Therefore $\frac{QT^2}{QR}$ is as $QT^2 \times SP^2$; that is, the *latus rectum* L is in the duplicate ratio of the area $QT \times SP$. Which was to be demonstrated.

Cor. Hence the whole area of the ellipsis, and the rectangle under the axes, proportional to it, is in the ratio compounded of the subduplicate ratio of the *latus rectum*, and the ratio of the periodical time. For, the whole area is as the area $QT \times SP$, which is described in a given time, multiplied into the periodical time.

PROPOSITION XV.—THEOREM VII.

The same things being supposed, I say, that *the periodical times in ellipses are in the sesquiplicate ratio of their greater axes.*

For the less axis is a mean proportional between the greater axis and the *latus rectum*; and, therefore, the rectangle under the axes is in the ratio compounded of the subduplicate ratio of the *latus rectum*, and the sesquiplicate ratio of the greater axis. But this rectangle (by Cor. Prop. XIV.) is in a ratio, compounded of the subduplicate ratio of the *latus rectum*, and the ratio of the periodical time. Subduct from both sides the subduplicate ratio of the *latus rectum*, and there will remain the sesquiplicate ratio of the greater axis equal to the ratio of the periodical time. Which was to be demonstrated.

Cor. Therefore, the periodical times in ellipses are the same as in circles, whose diameters are equal to the greater axes of the ellipses.

PROPOSITION XVI.—THEOREM VIII.

The same things being supposed, and right lines being drawn to the bodies, which touch the orbits; and perpendiculars being let fall on these tangents from the common focus: I say, that the velocities of the bodies are in a ratio compounded of the ratio of the perpen-

diculars inversely, and the subduplicate ratio of the principal latera recta *directly.*—(Fig. 8.)

From the focus S draw S Y perpendicular to the tangent P R, and the velocity of the body P will be reciprocally in the subduplicate ratio of the quantity $\frac{SY^2}{L}$. For that velocity is as the indefinitely small arc P Q described in a given moment of time; that is, (by Lem. VII.) as the tangent P R; that is, because of the proportionals P R to Q T and S P to S Y, as $\frac{SP \times QT}{SY}$, or as S Y reciprocally and $SP \times QT$ directly; but $SP \times QT$ is as the area described in a given time; that is, (by Prop. XIV.) in the subduplicate ratio of the *latus rectum*. Which was to be demonstrated.

Cor. 1. The principal *latera recta* are in a ratio compounded of the duplicate ratio of the perpendiculars, and the duplicate ratio of the velocities.

Cor. 2. The velocities of the bodies, in their greatest and least distances from the common focus, are in the ratio compounded of the ratio of the distances inversely, and the subduplicate ratio of the principal *latera recta* directly. For the perpendiculars are now the distances.

Cor. 3. And therefore the velocity in a conic section, at its greatest or least distance from the focus, is to the velocity in a circle at the same distance from the centre, in the subduplicate ratio of the principal *latus rectum* to double that distance.

Cor. 4. The velocities of bodies revolving in ellipses, at their mean distances from the common focus, are the same as those of bodies revolving in circles, at the same distances: that is, (by Cor. 6. Prop. IV.) reciprocally in the subduplicate ratio of the distances. For the perpendiculars are now the less semi-axes, and these are as mean proportionals between the distances and the *latera recta*. Let this ratio inversely be compounded with the subduplicate ratio of the *latera recta* directly, and we shall have the subduplicate ratio of the distances inversely.

Cor. 5. In the same figure, or even in different figures, whose principal *latera recta* are equal, the velocity of a body is reciprocally as the perpendicular let fall from the focus on the tangent.

Cor. 6. In a parabola, the velocity is reciprocally in the subduplicate ratio of the distance of the body from the focus of the figure: in the ellipsis it is more varied, and in the hyperbola less than according to this ratio. For (by Conics) the perpendicular let fall from the focus on the tangent of a parabola is in the subduplicate ratio of the distance. In the hyperbola the perpendicular is less varied; in the ellipsis more.

Cor. 7. In a parabola, the velocity of a body, at any distance from the focus, is to the velocity of a body revolving in a circle at the same distance from the centre, in the subduplicate ratio of the number 2 to 1; in the ellipsis it is less, and in the hyperbola greater, than according to this ratio. For (by Cor. 2. of this Prop.) the velocity at the vertex of a parabola is in this ratio, and (by Cor. 6. of this Prop. and Prop. IV.) the same proportion is preserved in all distances. And hence also in a parabola the velocity is every where equal to the velocity of a body revolving in a circle at half the distance; in an ellipsis it is less; in an hyperbola greater.

Cor. 8. The velocity of a body, revolving in any conic section, is to the velocity of a body revolving in a circle, at the distance of half the principal *latus rectum* of the section, as that distance, to the perpendicular let fall from the focus on the tangent of the section. This appears by Cor. 5.

Cor. 9. Since (by Cor. 6. Prop. IV.) the velocity of a body, revolving in this circle, is to the velocity of a body, revolving in any other circle, reciprocally in the subduplicate ratio of the distances; therefore *ex æquo* the velocity of a body, revolving in a conic section, will be to the velocity of a body revolving in a circle at the same distance, as a mean proportional between that common distance, and half the principal *latus rectum* of the section, to the perpendicular let fall from the common focus upon the tangent of the section.

FINIS.

NOTES,

&c.

TO THE

THREE PRECEDING SECTIONS.

GENERAL INTRODUCTION

TO THE

THREE SECTIONS.

Of the Method of Exhaustions.

Art. 1. BEFORE we enter upon the consideration of the doctrine of Prime and Ultimate Ratios, it may be of use to observe the steps by which the ancients were able, in several instances, from the mensuration of right-lined figures, to judge of such as are bounded by curve lines: *for as they did not allow themselves to resolve curvilinear figures into rectilinear elements,* it is worth while to examine by what art they could make a transition from the one to the other.

2. They found that similar triangles are to each other in the duplicate ratio of their homologous sides; and by resolving similar polygons into similar triangles, the same proportion was extended to these polygons also. But when they came to compare curvilinear figures, which cannot be resolved into rectilinear parts, this method failed. In these instances, they had recourse to what is called the Method of Exhaustions; the principle of which consisted, first, in describing upon the curvilinear space a rectilinear one, which, though not equal to the other, yet might

differ less from it than by any assignable quantity; and secondly, in investigating the truth or falsehood of every supposition that could possibly be made contrary to the proposition to be proved; and by reducing every such supposition to an absurdity, thence indirectly inferring the truth of the proposition itself. For instance, in comparing the areas of two circles, they inscribed in each similar polygons, which, by increasing the number of their sides, continually approached to the areas of the circles, so that the decreasing differences betwixt each circle and its inscribed polygon, by still further and further divisions of the circular arcs, could become less than any quantity that can be assigned: they found that all this while the similar polygons observed the same invariable ratio to each other, viz. that of the squares of the diameters of the circles. Upon this they founded their demonstration; and by shewing that some absurdity must follow if we suppose the circles to be to each other in a greater or in a less ratio than the squares of the diameters, they concluded that they must be in that very ratio. But as one complete instance may serve better than any general description, to exemplify their reasoning, let the following Theorem be proposed to be demonstrated by the method of Exhaustions.

3. *The area of a circle is equal to half the product of its radius and circumference.*—(Fig. 17.)

Let *b d*, the base of the right $\angle^{d}$. $\triangle$ *a b d*, be supposed equal to the circumference of the circle A B D, *a b* = radius C A, E F G H any equilateral polygon described about the circle, A B D K a similar polygon inscribed in it. As the circumscribed polygon E F G H is greater than the circle, so it is greater than the triangle *a b d* (being = to a $\triangle$ whose altitude is C A or *a b*, and base = perimeter E F G H, which is always greater than *b d*, the circumference of the ⊙). The inscribed polygon is less than the ⊙, and it is also less than the △ *a b d*, (being = to a △ whose altitude = C Q, which is less than C A or *a b*, and base = to its perimeter A B D K, which is less than the circumference *b d)* : ∴ the ⊙ and the △ *a b d* are both constantly limits betwixt the external and internal polygons E F G H, A B D K. ‖ Let the arc A B be bisected in L, and the tangent at L meet A E, B E in M and N, and the ∠ E L M being a right ∠, E M must be greater than L M or M A, the △ E L M greater than A L M, and E M N greater than the sum of the △s A L M, B L N, and consequently greater than half the space E A L B, bounded by the tangents E A, E B, and the arc A L B; ∴ (by Euclid 1. 10 B, the foundation of this method) the circumscribed polygon may approach to the ⊙ nearer than by any assignable quantity. ‖ The inscribed polygon may also approach to the ⊙ nearer than by any assignable quantity, as is shewn in the Elements of Euclid, ∴ the ⊙ and the △ *a b d*, which are both limits betwixt these polygons, must be equal to each other. For if the △ *a b d* be not = to the circle, it must either be greater or less than it. If the △ *a b d* be greater than the ⊙, then since the external polygon, by encreasing the number of its sides, may be made to approach the ⊙ so as to exceed it by a quantity less than any difference that can be supposed to exist between it and the △ *a b d*, it follows that the external polygon may become less than that △, which is absurd. If the △ *a b d* be less than the ⊙, then the inscribed polygon, by being made to approach the ⊙, may exceed that △, which is also absurd: Hence the circle and △ are equal to each other.

4. Archimedes in this demonstration does not suppose the circle to *coincide* with a circumscribed equilateral polygon of an infinite number of sides, but proceeds in a more accurate and unexceptionable manner.

And in this consists the error of many writers, who have asserted that curve lines were considered by the ancient geometers as polygons of an infinite number of sides. But this principle no where appears in their writings; we never find them resolving any figure or solid into infinitely small elements: on the contrary, they seem to avoid such suppositions, even when their demonstrations might have been sometimes abridged by admitting them. For instance, if they could have allowed themselves to have considered circles as similar polygons of an infinite number of sides; after proving that any similar polygons inscribed in circles are in the duplicate ratio of their diameters, they would have immediately extended this to the circles themselves. But there is ground to think, that they would not have admitted a demonstration of this kind. It was a fundamental principle with them, on which, as Archimedes expressly asserts, they founded their propositions on curvilinear figures, that the difference of any two unequal quantities may be added to itself until it exceed any proposed finite quantity of the same kind. But this principle seems to be inconsistent with the admitting of an infinitely small quantity or difference, which added to itself any number of times, is never supposed to become equal to any finite quantity whatsoever. The ancients, therefore, considered curvilinear areas as the limits of circumscribed or inscribed figures of a more simple kind, which approach to these limits, (by a bisection of lines or angles, that is continued at pleasure) so that the difference betwixt them may become less than any given quantity. The inscribed or circumscribed figures were always conceived to be of a magnitude, and N°. that is assignable; and from what had been shewn of these figures, they demonstrated the mensuration or the proportions of the curvilinear limits themselves, by arguments *ab absurdo.*

Of the Method of Indivisibles.

5. The doctrine of Exhaustions, as delivered by Archimedes, being considered tedious and prolix by the modern geometers, various methods were proposed for the purpose of simplifying and abridging his demonstrations. It was thought unnecessary to conceive the figures circumscribed about, or inscribed in, the curvilinear area or solid, as being always assignable and finite; and, therefore, instead of Archimedes' assignable finite figures, indivisible or infinitely small elements were substituted, and these being imagined indefinite or infinite in N°., their sum was supposed to coincide with the curvilinear area or solid.

6. It was upon these principles that Cavalerius, in the 17th century, founded what is called the Method of Indivisibles. In this doctrine, lines were conceived to be made up of an indefinite N°. of points, superficies of lines, and solids of superficies; and in computing the magnitudes or proportions of areas or solids, they computed the sum of all the indivisible elements of which the area or solid is composed. Thus for example, a △ was conceived to be made up of an indefinite N°. of lines parallel to the base, and consequently the area of the △ was equal to the sum of all these parallel lines. Now to find the sum of these parallel lines, we have only to conceive them as a set of quantities in arithmetical progression—the 1st term being 0, and the last term the base of the △, and the N°. of terms the perpendicular; ∴ the sum of the series, or the area of the △, will = base $\times \frac{1}{2}$ the perpendicular.

7. Ex. 2.—*To find the ratio betwixt the sphere and its circumscribing cylinder by the method of Indivisibles.*—(Fig. 18.)

Let the cylinder N M, the cone N O R, and the hemisphere M T S be cut by planes parallel to the base, one of which is C S K D C; then $SO^2 = CD^2 = SD^2 + DO^2 = SD^2 + DK^2$, $\therefore CD^2 = SD^2 + DK^2$; and this is true for every section parallel to the base: ∴ since the circles of which these line are the ½ diameters are as the squares of the said ½ diameters, it follows that the sum of all the circles in the ½ sphere, together with the sum of all the circles in the cone = the sum of all the circles in the cylinder: the cylinder itself ∴ which is composed of these circles is = to the ½ sphere and cone together; but the cone is a third part of the cylinder; this therefore being deducted, there remains ½ sphere : cylinder :: 2 : 3.

8. In this doctrine then we see, that by the admission of infinitely small quantities, the old foundation of geometry was abandoned, and suppositions resorted to which had been avoided by Archimedes. And though the new geometry had much the advantage over the ancient in point of conciseness; yet the former was much inferior to the other in the certainty of its deductions. For as this doctrine was inconsistent with the strict principles of geometry, so it soon appeared that there was some danger of its leading to false conclusions. And after men had indulged themselves in admitting quantities that were not assignable, and in supposing such things to be done as could not possibly be effected (against the constant practice of the ancients), and had involved themselves in the mazes of infinity, it was not easy for them to avoid perplexity, and sometimes error.

9. To shew the caution which should be used in the application of this doctrine, the following example may be sufficient. If a ⊙ be considered as a

polygon of an infinite number of sides, and ∴ an infinitely small arc be supposed perfectly to coincide with its chord, it follows that the time of the vibration of a pendulum in this arc = the time of descent down its chord;—but, by mechanics, the ratio of the times is that of the quadrant of a ⊙ to its diameter. Nor can this difficulty be removed except the arc be again divided into an infinite number of indivisible elements, infinitely less than the former; *i. e.* we must have recourse to infinitesimals of the 2d order.*

Of the Doctrine of Prime and Ultimate Ratios.

Art. 10. Having taken a general view of the ancient geometry, as it existed in the time of Archimedes, and the changes effected in it by the modern mathematicians, previous to Newton's time; we may now proceed to the consideration of the doctrine of Prime and Ultimate Ratios, which was invented by Sir I. Newton, for the purpose, as he himself says, of avoiding, on the one hand, the tedious demonstrations of the ancient, and on the other, the inaccurate and objectionable positions of the modern geometers.

* There is no such difficulty when the method of prime and ultimate ratios is applied to this case; for, though the arc and chord approximate to equality, the times of descending along them do not approximate; for, by the doctrine of limits, no part of a curve, how small soever, can ever be taken for a right line: but even when they so far approach to each other, that their lengths may be taken as equal, the curve still remains a curve; its inclination is different from that of the chord; the accelerating force along the curve perpetually varies, while the accelerating force along the chord remains constant, and consequently the times of describing these spaces are unequal, even supposing their lengths the same.

In this doctrine, magnitudes are not supposed to consist of indivisible parts, but to be generated by continued motion. Lineæ nempe (as Newton says) describuntur, ac describendo generantur, non per appositionem partium, sed per motum continuum punctorum; superficies per motum linearum, solida per motum superficierum, anguli per rotationem laterum, tempora per fluxum continuum, & sic in cæteris. Hæ geneses in rerum naturâ locum verè habent, & in motu corporum quotidiè cernuntur. This method of conceiving all variable quantities to be generated by motion is the characteristic feature, which distinguishes both this doctrine, and also that of fluxions.

11. This being premised, we now go on to the doctrine itself, the principle of which is contained in the following definition:—Let there be two quantities, one fixed, and the other varying, so related to each other that (1) The varying quantity, by a perpetual augmentation or diminution, continually approaches to the fixed quantity. (2) That the varying quantity does never pass beyond, or even actually reach that which is fixed. (3) That the varying quantity approaches nearer to the fixed quantity than by any assignable difference; then, upon the fulfilment of these three conditions, the fixed quantity is called the *Limit* or *Ultimate Magnitude* of the varying quantity.

12. *Ex.*—Take the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. the sum of which may be considered as continually varying, being perpetually increased by the accession of a new term; I say that the N°. 2 is the *limit* of the varying sum of the terms of this series. For (1) the varying sum continually approaches to the N° 2; the difference between the sum of one, two, three, four, &c. terms, and the N°. 2 being the N°s. 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. successively adinfinitum. (2) The sum can never exceed, or even become equal to 2; for no term in this series of differences can ever become either nothing

or negative. (3) We may continue the series till its sum approaches nearer to 2, than by any difference that can be assigned, as appears from the terms of the series of differences, which may be continued till they become less than any assignable quantity. The N°. 2 then, having the conditions laid down in Art. 11, is the *limit* of the sum of the infinite series.

13. The explanation given in Art. 11., of quantities which have limits, is also to be extended to the limits of ratios. The definition may be thus stated. If there be two quantities that are (one or both) continually varying, either by being continually augmented, or continually diminished; and if the ratio they bear to each other does, by this means, perpetually vary, but in such a manner, that (1) this varying ratio continually approaches to some determined ratio; (2) that the varying ratio does never pass beyond, or even actually reach, the fixed ratio; (3) that the varying ratio approaches nearer to the fixed ratio than to any other that can be assigned: then, upon the fulfilment of these three conditions, the determined ratio is called the *limiting* or *ultimate* ratio of the varying one.

14. *Ex.*—Take the ratio $3x + 4 : 2x + 1$, where both terms are variable, by the variation of x; then if x decrease in infinitum, I say that the determined R°. $4 : 1$ is the *limiting* R°. of the variable proportion $3x + 4 : 2x + 1$. For (1) as x decreases, the quantities $3x$ and $2x$ decrease, and consequently the R°. $3x + 4 : 2x + 1$ approaches to that of $4 : 1$: (2) The R°. $3x + 4 : 2x + 1$ can never exceed, or even reach, that of $4 : 1$; for $8x + 4 : 2x + 1 :: 4 : 1$, but $8x + 4$ is always greater than $3x + 4$; $\therefore 3x + 4 : 2x + 1$ is always in a less R°. than that of $4 : 1$; (3) The Ratio $3x + 4 : 2x + 1$ will approach nearer to that of $4 : 1$ than to any other that can be proposed; for in the terms of this R°. $3x + 4 : 2x + 1$, the varying parts $3x$ and $2x$, by diminishing x, may become less than any

assignable quantity; ∴, by Art. 13, the R°. 4 : 1 is the limiting R°. of $3x + 4 : 2x + 1$.

15. In the last Ex. we may observe that, though x is supposed to decrease in infinitum, yet the *terms* of the R°. no less than the R°. itself, always continue finite. But this is not a necessary condition; for a R°. may never vary beyond certain limits, even though the terms themselves should increase or decrease in infinitum; and since Ratios of this sort are of most frequent recurrence in this doctrine, and peculiarly characteristic of it, we will now proceed to the consideration of them.

16. *Ex.* 1.—Let x be any varying quantity; make $4x^2 + 3x = A$, and $2x^2 + x = B$, then will A and B also be varying quantities, as depending upon x; when x vanishes, A and B will both vanish; and when x is infinite, they will both be infinite: I say, that the determined R°. 3 : 1 is the limiting R°. of A : B, while x decreases in infinitum. For the R°. A : B = the R°. $4x + 3 : 2x + 1$; ∴ (1) as x decreases, A : B approaches to the R°. 3 : 1; (2) the R°. A : B can never exceed, or even reach, that of 3 : 1; for $6x^2 + 3x : 2x^2 + x :: 3 : 1$, but $6x^2 + 3x$ is greater than $4x^2 + 3x$; ∴ $4x^2 + 3x$ is always in a less R°. to $2x^2 + x$ than the R°. 3 : 1; (3) the Ratio A : B will approach nearer to that of 3 : 1, than to any other that can be proposed; for $4x$ and $2x$ may become less than any assignable quantity, by the diminution of x; consequently the R°. 3 : 1 is the limiting R°. of $4x^2 + 3x : 2x^2 + x$.

Ex. 2.—Taking the same R°. as before; I say, that while x increases in infinitum, the determined R°. 2 : 1 is the limiting R°. of A : B; for the given R°. = that of $4 + \frac{3}{x} : 2 + \frac{1}{x}$; ∴ (1) the Ratio A : B approaches that of 2 : 1; for as x increases $\frac{3}{x}$ and $\frac{1}{x}$ decrease; (2) The R°. A : B can never be less than, or even equal to, the R°. $2 : 1$; for $4x^2 + 2x : 2x^2 + x :: 2 : 1$; ∴ $4x^2 + 3x$ is always to $2x^2 + x$ in a greater R°. than that of $2 : 1$; (3) The R°. A : B will approach nearer to that of $2 : 1$ than to any other that can be proposed; for $\frac{3}{x}$ and $\frac{1}{x}$, by increasing x, may become less than any assignable quantity; consequently the R°. $2 : 1$ is the limiting R°. of $4x^2 + 3x : 2x^2 + x$.

17. We see then in the two last Examples, that though diminishing x, and consequently diminishing the terms A and B, increases their R°.; and contrariwise increasing these terms, by increasing x, decreases their R°.; yet there is a limit both to the increase and decrease of this R°., though there is none to the terms themselves that compose it, which, as we have seen, in the first case decrease, and in the other increase, in infinitum.

18. We will close these Examples, by proposing a geometrical one of the same kind with that given in Art. 16, which is added for the purpose of more clearly explaining Newton's phrases of "Ratio ultima quantitatum evanescentium," and "Ratio prima Quantitatum nascentium." Let *(Fig. 19)* A B C D, E B C F be two quadrilateral figures, and let D F be parallel to A E; then the quadrilateral A B C D bears to the quadrilateral E B C F the proportion of A B + D C to E B + C F. Now if the line D F be supposed to advance towards A E, with an uninterrupted motion, till the quadrilaterals quite disappear or vanish, this proportion of A B + D C : B E + C F will, during this motion, continually vary, (unless the lines D A, C B, F E produced meet in the same point, which they are not here supposed to do) and this proportion, by diminishing the distance between D F and A E may at last be brought nearer to the proportion of A B : B E than to any other whatever; though it can never exceed, or even actually

reach, this proportion; ∴ the proportion of A B : B E is the limiting or ultimate proportion of the quadrilateral A B C D: the quadrilateral E B C F, bcause it is the proportion which these quadrilaterals can never actually have to each other, but the limit of that proportion.

In this Ex. then, as in the other above given, the quantities themselves, *i. e.* the quadrilaterals, have neither of them any final magnitude, or even so much as a limit; but, by the diminution of the distance between D F and A E, diminish continually without end; yet there is a limit to the varying proportion existing between them, viz. that of A B : B E; and hence this limit is to be called the *ultimate* R°. of the *vanishing* quadrilaterals.

19. But that the meaning of the expression "Ratio ultima quantitatum evanescentium" may be still more clearly understood, we may further observe (1) That since the quadrilaterals diminish by a continual motion till they actually vanish, they may properly be called *vanishing* quantities; since under this view they have never any stable magnitude, but decrease by a continued motion till they become nothing. (2) That the quadrilaterals A B C D, B E F C, become vanishing quantities, from the time we first ascribe to them this perpetual diminution, *i. e.* from that time they are quantities going to vanish. And as during their diminution they have continually different proportions to each other; so the R°. between A B and B E is not to be called merely Ratio harum quantitatum evanescentium; but *ultima* Ratio, &c. and the same observations are applicable to the Example given in Art. 16.

20. Should we suppose the line D F first to coincide with the line A E, and then to recede from it, thus giving birth to the quadrilaterals; then under this conception, the R°. A B : B E, as it was before called the R°. wherewith the quadrilaterals *vanish*, is now to be considered as the R°. wherewith the qua-

drilaterals by this motion *commence*; and the R°. may also properly be called the *first* or *prime* R°. of these quadrilaterals *at their origin*.

21. As in Art. 19, the phrase *vanishing* quantities was applied to the quadrilaterals, from the time that they are quantities going to vanish; so, under the present conception, they are to be called *nascentes*, not only at the very instant of their first production, but according to the sense in which such participles are used in common speech; just as when we say of a body, which has lain at rest, that it is beginning to move, though it may have been some little time in motion. On this account we must not use the simple expression, Ratio quantitatum nascentium, but Ratio *prima* quantitatum nascentium.

22. We see here the same R°. may be called sometimes the *Prime*, at other times, the *Ultimate*, R°. of the same varying quantities, according as these quantities are considered under the notion of vanishing, or of being produced, before the imagination, by an uninterrupted motion. The doctrine under examination receives its name from both these ways of expression.

23. There are two objections to this method proposed and answered by Newton in his Scholium to the 1st Section, which it may be worth while briefly to notice; though they may have been sufficiently obviated, the first in Art. 18, and the second in Articles 19 and 21. The first objection states, that there is no ultimate proportion of vanishing quantities, forasmuch as before they vanish, the proportion is not the ultimate proportion; and when they have vanished, it is nothing. But Newton observes, that it might with equal justice be contended that there is no ultimate velocity of a body falling by gravity to the earth's surface, inasmuch as before it has reached the earth, the velocity is not the *ultimate* velocity; and after it has reached the earth, it is nothing. The answer in both cases is easy, when the meaning of the term *ultimate* is

carefully kept in view. By the ultimate velocity then is to be understood that, with which the body moves, neither before it arrives at the earth, nor after; but that very velocity with which it arrives: so by the ultimate R°. of vanishing quantities is meant the R°. of the quantities, not before they vanish, nor after; but that with which they do vanish. In like manner, by the prime R°. quantitatum nascentium, is meant the R°. with which they start into existence: there exists a *limit* to the velocity in the one case, and to the varying R°. in the other; and this limit, as has been frequently observed, is all that is meant by the term ultimate proportion.

In the second objection it is contended, that if the ultimate Ratios of vanishing quantities be given, the ultimate or vanishing quantities themselves will be given; *i. e.* the quantities themselves will have attained a limit to their decrease, which they cannot pass; and thus every quantity will consist of indivisible parts. If by the term ultima ratio quantitatum evanescentium were meant the ratio of ultimate quantities, the objection might have some weight; for then it might be inferred from the expression, that these ultimate quantities had attained some final magnitude; but Newton never supposes this: on the contrary, by ultimate or evanescent quantities he means quantities, to the decrease of which there is no limit; and consequently, by the ultimate ratios quantitatum evanescentium, is to be understood, not the Ratios of ultimate magnitudes, but, as we have seen, the limit of the Ratios of quantities decreasing without end; to which limits the varying Ratios may approach nearer than to any other that can be assigned, but which they can never pass, nor even equal, till the quantities are diminished in infinitum: just as when two quantities, whose difference is given, increase in infinitum, their ultimate R°. is given, viz. a R°. of equality; and yet the ultimate quantities themselves are not given, because they can never reach their ultimate or maximum state. Newton therefore cautions his readers, if at any time he should use the words "quantitates quam minimæ, vel evanescentes, vel ultimæ," not to understand quantities determined in magnitude (how small soever), but quantities, to the decrease of which there is no limit. And herein the method of Prime and Ultimate Ratios essentially differs from that of indivisibles; for in that method these ultimate quantities are considered absolutely as parts, whereof their respective quantities are actually composed. But though these ultimate quantities have no final magnitude, which would be necessary to make them parts capable of compounding a whole by accumulation, yet their ultimate Ratios are as truly assignable as the Ratios between any quantities whatever. Therefore none of the objections made against the doctrine of Indivisibles, are of the least weight against this method.

NOTES TO SECTION I.

LEMMA I.

24. *Case* 1:—Let there be two variable quantities x and y, which continually approach to equality, so that their difference, when compared with either of them, becomes at length less than any assignable quantity; then will x and y be ultimately equal: in other words, if a be the ultimate magnitude of x, and b the ultimate magnitude of y, these limits a and b will be accurately equal. For if not, let these limits have a difference, d, *i. e.* let $b = a + d$; then since a is the limit of x, x can never exceed a, and ∴ can never come nearer to $a + d$, the limit of y, than by the given difference d; *i. e.* x and y, even in their ultimate state, can never approach nearer to each other than by the given difference d; which is contrary to the hypothesis: ∴ a does accurately $= b$, *i. e.* x and y are ultimately equal. Here x has been supposed to be less than its limit a; but the Prop. may be proved after the same manner, if x be supposed to be greater than a.

Case 2. Let there be two variable Ratios $x : y$ and $v : z$, which continually approach to equality; so that at length the R°. $x : y$ approaches nearer to that of $v : z$ than to any other that can be assigned; then will the R°. $x : y$ be ultimately = the R°. $v : z$; in other words, if $m : n$ be the limiting R°. of $x : y$, and $p : q$ the limiting R°. of $v : z$, the R°. $m : n$ shall accurately = that $p : q$. For if not, let there be any given difference between them; then, since the Ratios $x : y$ and $v : z$ can never actually reach their limits $m : n$ and $p : q$; it follows that $x : y$ and $v : z$ can never approach nearer to equality than by this given difference, which is contrary to the hypothesis; ∴ the R°. $m : n$ does accurately = that of $p : q$; i. e. the Ratios $x : y$ and $v : z$ are ultimately equal.

Or both cases may be concisely proved, by observing, that both quantities, and the Ratios of quantities, such as are understood in the Lemma, cannot approach nearer to each other than their limits do; and the absurdity of supposing these limits unequal is immediately apparent.

LEMMA III.

Note to Lemma 3.

25. What is here proved of the *areas* of the inscribed and circumscribed figures is not true of the *perimeters*; for the ∠r boundary of the circumscribed always remains the same, being = A a + A E, whatever be the N°. of divisions; and ∴ never approaches the curvilinear boundary as a limit; and the ∠r boundary of the inscribed approaches that of the circumscribed as a limit, and is always greater than the curvilinear boundary. Hence Newton's *ultimate sum* in Cor. 1. must be strictly confined to area.

Lem. 3.—Cor. 3.

26. For (*Fig.* 20) one of the lines at least in each pair *a l*, *l b*, *b m*, *m c*, *c n*, *n d*, must cut the curve, consequently one of the lines at least in each pair must make a greater ∠ with the curve than the tangents do; hence the △s. *a p b*, *b o c*, *c r d*, formed by the tangents, will fall within the mixtilinear spaces *a l b*, *b m c*, *c n d*, and ∴ be less than them; consequently since A*a l b m c n d* is ultimately = the curvilinear area, much more will the area A*apbocrd* be ultimately = the same curvilinear area.

Notes to Lem. 3.—Cor. 4.

27. The *ultimate figures* here spoken of, must be applied only to the figures of the chords and tangents, since the ∠r perimeters above mentioned, have not the curve line for their limit. The Cor. so far as relates to the chords, is perfectly evident; if the reader should not think it equally so for the figure formed by the tangents, he may see a proof of it in Art. 37.

28. *Curvilinear limits of rectilinear figures.* See Scholium to Lemma XI. where Newton again cautions his readers,* that if at any time he should, for right lines, substitute curve lineolæ, they are not to understand that these lineolæ are made up of right lines, however small, (agreeably to the doctrine of Indivisibles) but that the curves are the limits, to which the vanishing right lines continually approach, and ultimately equal.

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* "Si pro rectis usurpavero lineolas curvas, nolim indivisibilia, sed evanescentia divisibilia."

LEMMA IV.

29. For, by hypothesis, $A' : a' :: B' : b' :: C' : c'$ ultimately; $\therefore A' : a' :: A' + B' + C' : a' + b' + c'$ ultimately; but ultimately $A' + B' + C' =$ whole figure D E F, and $a' + b' + c' =$ whole figure def; $\therefore$ under the conditions mentioned in the Lemma, D E F : $d e f$ in the given R°. of $A' : a'$.

LEMMA V.

Introductory Articles to Lemma 5.

30. *Definition.*—Curvilinear figures are said to be similar, when they may be supposed to be placed in such a manner, that any right line being drawn from a determined point to the terms that bound them, the parts of the right line, intercepted betwixt that point and those terms, are always in one constant R°. to each other. Thus the curvilinear figures A S D, a S d, (*Fig.* 21) or the figures S P D, Sp d are similar, when any line S P being drawn always from the same point S, meeting the two curves in P, p, the R°. of S P : Sp is invariable.

31. It follows, from this definition, that if there be two similar curvilinear figures, and any rectilinear figure be inscribed in one, a similar rectilinear figure may be inscribed in the other. For let A S D, $a s d$ (*Fig.* 22) be two similar curvilinear figures, and let any rectilinear figure whatever, S A P Q D, be inscribed in one of them, S A D; I say, that a similar rectilinear figure may be inscribed in the other $s a d$: from s draw $s p$, $s q$, &c. making the $\angle^s$ $a s p$, $p s q$, &c. = the $\angle^s$ A S P, P S Q, &c.; and consequently

the remaining $\angle\ qsd$ will = the remaining $\angle$ QSD, join ap, pq, qd, &c.; then since S A : sa :: S P : sp (by definition) ∴ S A : S P :: sa : sp, and $\angle$ A S P = $\angle\ asp$; ∴ △ˢ A S P, asp are similar, and the same may be shewn of all the remaining △ˢ; ∴ the polygon $sapqd$ is similar to the polygon S A P Q D.

32. And hence it is, that this last property has been frequently made the criterion of similar curvilinear figures; *i. e.* curvilinear figures have been defined to be similar, when, any rectilinear figure being inscribed in one, a similar rectilinear figure may be inscribed in the other; which being the case, the definition above given must be *proved*, to follow as a consequence from the latter, thus:

Let S A D, sad be two similar curvilinear figures; in S A D inscribe any polygon *whatever* S A P Q D; then, since the figures are similar, a similar polygon may, by the definition, be inscribed in the other sad: let $sapqd$ be this polygon; consequently these polygons may (by Euclid) be divided into the same N°. of similar △ˢ: let them be so divided; then since the △ˢ S A P, sap are similar, as also the △ˢ SPQ, spq, and S Q D, sqd; S A : S P :: sa : sp; S P : S Q :: sp : sq; and S Q : S D :: sq : sd; ∴ S A : sa :: S P : sp :: S Q : sq, &c.; *i. e.* these lines S A, sa; SP, sp, &c. are to each other in a constant Ratio.

Hence either of the above properties may be assumed as the definition of similar curvilinear figures, since they are each mutually deducible from, and consistent with, the other. The last definition may be as convenient in the following Lemma; but in the remaining ones the first may be used with advantage.

Lemma 5.

33. (1) Let S A D, sad (last Fig.) be two similar curvilinear figures, and let S A P Q D be any polygon

inscribed in the former, and *s a p q d* a similar polygon inscribed in the latter (Art. 32); then since the polygons are similar A P : P Q :: *a p* : *p q*; and P Q : Q D :: *p q* : *q d*; ∴ A P : *a p* :: P Q : *p q* :: Q D : *q d*, &c., and this is true when the N°. of the sides A P, *a p*, P Q, *p q*, &c. is increased, and their magnitude diminished without limit; ∴ (by Cor. Lem. IV.) curve A P D : curve *a p d* :: A P : *a p* :: S A : *s a*.

(2) Taking the same construction as before, since polygons S A P Q D, *s a p q d* are similar, the △s into which they are divided will be similar; ∴ △ S A P : △ *s a p* :: △ S P Q : △ *s p q* :: △ S Q D : △ *s q d*, &c.; ∴ as before, curvilinear area S A D : curvilinear area *s a d* :: △ S A P : △ *s a p* :: SA^2 : sa^2.

LEMMA VI.

Introductory Articles to Lemma 6.

34. A curve of *continued* curvature may be defined to be a line traced out by a point, *continually* changing its direction; where we may observe that the word *continually* implies that the change of direction of the generating point must not be effected by starts or impulses *(per saltum)*, but by an uninterrupted and equable motion. Thus the ∠ B C D, *(Fig. 23)* which measures the variation of direction of the generating point at A and B, (while the point moves from B to A) must, before it become nothing, pass through all the intermediate degrees of magnitude, from B C D to nothing.

From this definition it will appear that two curves which cut one another, as E *d*, *d* F, *(Fig. 24)* cannot be called a curve of continued curvature at the point *d*; for if *a* and *c* be taken on opposite sides of *d*, the variation of direction from *a* to *c*, viz. the ∠ *c b g* has been effected *per saltum*; *i. e.* in passing from nothing to *c b g*, the ∠ has not passed through all the intermediate degrees of magnitude.

35. From hence also it follows, (1) That if the distance betwixt two positions of the generating point continually decrease, and at length ultimately vanish, the change of direction of this point will also continually decrease, and at length ultimately vanish; *i. e.* while B moves up to A (*Fig.* 23) the ∠ B C D is decreasing continually without limit, till at last, when A B ultimately vanishes, the ∠ B C D also ultimately vanishes. (2) That the direction of the generating point is a tangent to the curve; for, suppose A D to be the direction of the generating point at A, then, if it did not change its direction, it would move along the line A D; but, by the definition, it is continually changing its direction; ∴ if it be in the line A D at A, it will not continue in it, but will, in the next moment of time, go either above or below it; ∴ A D is a tangent to the curve at A. (3) That A D is the only tangent; for, if possible, let A V, (*Fig.* 25) making a finite ∠ with A D, be a tangent, let the point B move up to A, so that the change of direction B C D may be indefinitely small, then will B C D be indefinitely less than D A V; ∴ a fortiori will the interior ∠, formed by the curve and tangent D A, be indefinitely less than D A V; *i. e.* D A passes indefinitely nearer the curve than any other line A V that can be drawn.

Lemma 6.

36. After what has been premised, the Lemma may be easily proved thus. Let A, B (*Fig.* 25) be two positions of the generating point, draw the chord A B, and at the points A, B, draw A C, B C in the

direction of the generating points at A and B respectively; then A C, B C are tangents to the curve, (Art. 35.) Now, by the continual approach of B to A, the change of direction of the generating point will continually decrease, and at length ultimately vanish, (Art. 35) *i. e.* the ∠ B C D will ultimately vanish; a fortiori ∴ will the interior ∠ B A D, contained by the chord and tangent, ultimately vanish.

Note to Lemma 6.

37. By the help of this Proposition, Cor. 4. Lem. III. may be easily proved. Let the two lines A D, D B, (*Fig.* 26) which touch the curve A C B of continued curvature in the points A, B meet each other in D, and the chord A B be drawn; the sum of the tangents will be greater than the chord; and if the chord be divided into any two parts in the point C, and the chords A C, C B be drawn, and also E F a tangent to the curve in the point C, meeting the tangents A D, B D in E and F, the sum of the chords A C, C B will be greater than the first chord A B; and the sum of the tangents A E, E C, C F, F B, greater than the sum of the chords: but A E, E F being less than A D, D F; A E, E F, F B will be less than A D, D B. Hence, if the N°. of parts, into which the curve A C B is divided, be continually increased, the sum of the chords will be continually increased, and the sum of the tangents continually diminished; and the latter sum being always greater than the former, the difference between them will continually decrease; and as the ∠^{s} between the chords and tangents way be diminished mithout limit, (Art. 36) this difference may be also diminished without limit. Hence the difference between the perimeters of the figures, contained by the two lines A *a*, A E, (*Fig.* 1) and the chords, and by the same two lines and the tangents, will be continually diminished, as the bases A B, B C, C D, &c. are diminished; and

the perimeter of the curvilinear figure will be a limit to them both.

LEMMA VII.

Introductory Article to Lemma 7.

38. It follows from the definition of similar curvilinear figures given in Art. 30, (1) that to draw a curve A *c b* similar to another A C B (*Fig.* 27), we must produce A B to any point *b*, and, while A *b* revolves round A as a centre, let the point *b* move in the line A *b*, so that A *b* may be to A B in a given R°.; then will A *c b* be similar to A C B: (2) that if A D be a tangent to A C B at A, it will also be a tangent to the similar curve A *c b* at A; for draw *b d* parallel to B D, then by similar $\triangle^s$, *b d* : B D :: A *b* : A B, in a given R°.; ∴ *b d* will not vanish till B D vanishes, *i. e.* at the point A.

Lemma 7.

36. Produce A D (*Fig.* 27) to any distant point *d*, and let *d b* be drawn parallel to D B, meeting the chord A B produced in *b*; and through the point *b* describe, as has been above shewn, the curve A *c b* continually similar to A C B, to which A *d* will be a tangent; then, by similar $\triangle^s$, A B : A D :: A *b* : A *d*; and by similar figures (Lem. 5.) A C B : A *c b* :: A B : A *b*, or as A D : A *d*; ∴ the chord, arc, and tangent A B, A C B, and A D are always proportional to the chord, arc, and tangent A *b*, A *c b*, and A *d*. But when B moves up to A, the ∠ *b* A *d* (= ∠ B A D) will, by Lem. 6, ultimately vanish; ∴ A *b*, and also the intermediate arc A *c b*, will continually approach A *d*, and at length will ultimately coincide with, and become equal to it; and conse-

quently A B, A C B, and A D, which are always proportional to these, will also ultimately be to each other in a R°. of equality.

Notes to Lemma 7.

40. In the demonstration B D is supposed to move parallel to itself, as B moves up to A, while *b d* remains fixed. Hence (1) by the motion of B towards A, A *b* is continually approaching nearer to A *d* without limit; while, at the same time, it carries the intermediate arc A *c b* (which is continually unbending itself) along with it. (2) The *magnitudes* of A *b* and A *c b* also continually approach to that of A *d*, nearer and nearer without limit; though these quantities can never exceed A *d*, nor indeed equal it, till B and A actually coincide; ∴ the *finite* lines A *b*, A *c b*, and A *d* ultimately coinciding are equal; whence this is also inferred of the *vanishing* lines A B, A C B, and A D, which are always proportional to them.

41. The Lemma is frequently explained by supposing R B D *(Fig.* 3) to move round R fixed as a centre, while, by this revolution, B continually approaches to A; at the same time *d r* moves round the fixed point *d* in a contrary direction, so as always to keep parallel to R B D. But this explanation is clearly at variance with Newton's notions, as is evident from the next Lemma.—See Art. 44.

42. Since it would be difficult for the understanding, in contemplating quantities, which elude the notice of the senses, clearly to perceive the changes which take place in the vanishing chord, arc, and tangent, and the limit to which their proportions continually approach, Newton has had recourse to the artifice of substituting, in the room of these *vanishing* quantities, *finite* ones, which bear a constant proportion to the others; and by ascertaining the limit which the R°. between the latter ultimately attains, on the coincidence of B and A, he discovers also the limit of the Ratios of the vanishing quanti-

ties, which are proportional to them. The same observation is applicable to the 8th and 9th Lemmas.

LEMMA VIII.

43. Produce A D to any distant point *d*, and draw *d b r* parallel to D B R, meeting A B and A R produced in *b* and *r*; and through *b* describe the curve A *c b* always similar to A C B; then the figures R A B, R A C B, and R A D are always similar to *r* A *b*, *r* A *c b*, and *r* A *d*; they are likewise always proportional to them. For R A B : *r* A *b* :: R A^2 : *r* A^2 :: R A D : *r* A *d*; ∴ R A B : R A D :: *r* A *b* : *r* A *d*; also sector R A C B : sector *r* A *c b* :: R A^2 : *r* A^2 (Lemma IV.) :: R A D : *r* A *d*; ∴ R A C B : R A D :: *r* A *c b* : *r* A *d*. Now let B move up to A, and ultimately coincide with it, then the ∠ *d* A *b* (= ∠ D A B) will ultimately vanish; ∴ the three continually finite △s *r* A *b*, *r* A *c b*, and *r* A *d* will ultimately coincide with each other, and consequently be ultimately similar and equal to each other: ∴ also the vanishing △s R A B, R A C B, and R A D, which are always similar and equal to the former, will also be ultimately similar and equal to each other.

Note to Lemma 8.

44. It is plain, from the words "triangula tria *semper finita*," in this Lemma, that R B D is supposed to move parallel to itself, while *d b r* remains fixed; and not that R B D moves round R as a fixed point; for in the latter case the △s *r* A *b*, *r* A *c b*, *r* A *d* would be ultimately infinitely great, and the purpose for which these last △s were introduced (see Art. 42.) thus rendered useless.

LEMMA IX.

45. Produce A E to any distant point e, and take $Ae : Ad :: AE : AD$; draw ec, db parallel to E C, D B, and let them meet the chords A C, A B produced in c, b; then the △s Adb, Aec being similar to A D B, A E C respectively; $Ab : AB$ ($:: Ad : AD :: Ae : AE$) $:: Ac : AC$; ∴ c, b, will be in the curve Abc, which is similar to A B C; in the same manner during the approach of C and B to A, the points b, c, determined in like manner, will always be found in a curve similar to A B C; and because the curves Abc, A B C are similar, the areas Abd, Ace will be similar to the areas A B D, A C E respectively, and they are ∴ proportional to each other respectively; for $ABD : Abd$ ($:: AD^2 : Ad^2 :: AE^2 : Ae^2$) $:: ACE : Ace$; ∴ altern°. $ABD : ACE :: Abd : Ace$. To the similar curves A B C, Abc draw the tangent $AFGfg$; then as C and B move up to A, and ultimately coincide with it, the ∠ cAg is continually diminished, and will ultimately vanish, ∴ the curvilinear areas Abd, Ace will ultimately coincide with the rectilinear areas Afd, Age; and be ∴ ultimately to each other as $Ad^2 : Ae^2$; ∴ also will the curvilinear areas A B D, A C E, which are proportional to these others, be also ultimately in the Ratio of $Ad^2 : Ae^2$ or of $AD^2 : AE^2$.

Note to Lemma 9.

46. We may observe here, that the ∠, which E A makes with the curve, as indeed all *determined* ∠s, and quantities of whatsoever kind in this and the following Sections, are supposed to be finite; Newton disclaims the use of infinitely small *determinate* quantities as unintelligible, and by the words infi-

nitely small $\angle^s$, or infinitely small quantities, he means *variable* quantities, which by a continual flux are decreasing without limit, (see Art. 23.)

LEMMA X.

Introductory Article to Lemma 10.

47. If the abscissæ A B, A D, (*Fig.* 28) be as the times in which a body, urged by any finite force, describes two spaces; and the ordinates B C, D E be as the velocities generated in those times; and if A C E be the curve traced out by the extremities of these ordinates, the areas A B C, A D E will be as the spaces described.

Let the times be divided into any N°. of equal parts A F, F G, G H, &c., and complete the parallelograms A K, F L, G M, &c.; then if the force be supposed to act only at equal intervals of time, so as to make the body move uniformly during the times A F, F G, G H, &c. with the velocities F K, G L, H M, &c., the spaces described in these times will be represented by the parallelograms, and the sums of the spaces by the sums of the parallelograms. Now let the intervals of time be continually diminished, then will the force, which now acts by impulses, continually become nearer and nearer a force acting incessantly; and the sums of the parallelograms, which represent the spaces, continually approach nearer and nearer to the curvilinear areas, till at length, when the intervals of time are diminished, and their N°. increased in infinitum, the force will become an incessant force, and at the same instant the sums of the parallelograms become = the curvilinear areas (Lem. II.); ∴ under the circumstances mentioned in the Proposition, the spaces will be accurately measured by the curvilinear areas.

We may observe that in this, and Propositions of the like nature, a false hypothesis is made, viz. that the force acts by impulses, and by consequence we deduce a false conclusion, viz. that the spaces are represented by the sums of the parallelograms; but as the assumed hypothesis approaches to the true, so does the false conclusion approach to the true conclusion; till at length, upon the attainment of the true hypothesis, we attain at the same time the true conclusion: the true hypothesis and true conclusion being respectively the limits of the assumed hypothesis, and the conclusion consequent upon it.

Lemma 10.

48. Let the times be represented by the lines A D, A E, and the velocities generated, by the ordinates D B, E C, then the spaces described with these velocities will, by what has been just proved, be represented by the areas A B D, A C E described by these ordinates; but the prime R°. of these nascent areas A B D, A C E is (Lem. IX.) that of $AD^2 : AE^2$; *i. e.* the spaces described are, in the very beginning of the motion, in the duplicate R°. of the times in which they are described.

Lem. 10.—*Cor.* 1.

49. Let A B and *a b* (*Fig.* 30) be similar parts of similar figures described by two bodies in proportional times; and let two equal forces similarly applied act upon the bodies, sufficient to make them move from B to C, and from *b* to *c*, in the time that they would have described A B, *a b*; then they will describe two other curves A C, *a c*; and the limiting R°. of B C : *b c* (which, as being the distances the bodies have erred from their former course, are called errors in this Corollary) will be that of the squares of the times in which A B, *a b* would have been described. For

B C, *b c* may be considered as spaces described from rest in those times by equal forces, and ∴ the Lemma is applicable to them.

Note to Lemma 10.—*Cor.* 1.

50. "Are nearly, &c."—Though strictly speaking, by the spaces mentioned in this Lemma are meant not any spaces actually described, however small they be taken, but only the limiting ratio of the spaces; yet still if B C, *b c* be *actual* spaces described, provided they are sufficiently small, they will be as the square of the times quam proxime, *i. e.* without any sensible error; and thus this and the next Corollary are applied in the 66th Proposition to find the errors produced in the motions of the moon, &c. by the attraction of the sun.

Lemma 10.—*Cor.* 3.

51. Let A D, *a d (Fig. 29)* represent two *equal* times, D B, *d b* the velocities generated in those times; then will the spaces be represented in the two cases by A D B, *a d b*; but A D B : *a d b* :: A D × D B : *a d* × *d b* ultimately, :: D B : *d b* ultimately (since A D = *a d*) *i. e.* in the very beginning of the motion, space described varies as the momentary increment of velocity when the time is given; but the velocities generated in an indefinitely small given time are proper measures of the accelerating forces; ∴ in the very beginning of the motion, space varies as force, when time is given; but (by Lemma) space varies as T^2, when force is given, ∴ when neither are given, the space will, in the very beginning of the motion, vary as $F \times T^2$.

LEMMA XI.

Introductory Articles to Lemma 11.

52. Any two arcs of curve lines touch each other

when the same right line is the tangent of both at the same point; but when they are applied upon each other they never perfectly coincide, unless they are similar arcs of equal and similar figures; and the curvature of lines admits of an indefinite variety. Because the curvature is uniform in a given ⊙, and may be varied at pleasure in them, by enlarging or diminishing their diameters, the flexure or curvature of circles serves for measuring that of other lines.

53. As of all the right lines, that can be drawn through a given point in the arc of a curve, that is the tangent which touches the arc so closely, that no right line can be drawn between them; so of all the circles that touch a curve in any given point, that is said to have the *same curvature* with it, which touches it so closely that no ⊙ can be drawn through the point of contact between them; all other circles passing either within or without them both. This ⊙ is called the ⊙ of curvature belonging to the point of contact. The arc of this ⊙ cannot coincide with the arc of the curve, but it is sufficient to denote it the ⊙ of curvature that no other ⊙ can pass between them; as the tangent of the arc of a curve cannot coincide with it, but is applied to it so that no right line can be drawn between them. As in all curvilinear figures the position of the tangent is continually varying, so the curvature is continually varying in all curvilinear figures, the ⊙ only excepted. As the curve is separated from its tangent by its flexure or curvature, so it is separated from its ⊙ of curvature in consequence of the encrease or decrease of its curvature: and as its curvature is greater or less, according as it is more or less inflected from the tangent, so the variation of curvature is greater or less, according as it is more or less separated from the ⊙ of curvature. It is manifest that there is but one ⊙ of curvature belonging to an arc of a curve at the same point; for if there were two such circles, any circles described between these through that point would pass between

the curve and ⊙ of curvature, against the supposition. Having thus shewn what the ⊙ of curvature is; it will be necessary to point out, in the next place, the method of describing it; this is done by the following proposition:—

54. *Let* E M H (Fig. 31) *be any curve,* E T *a tangent at the point* E, E B b *a right line, making any* ∠ *with* E T; T M R *any straight line parallel to* E B, *meeting the tangent in* T, *and the curve in* M; *then if the rectangle* M T × T K *be always taken* $= ET^2$, *and* F K B *be the curve traced out by the point* K, *thus taken, and if this curve ultimately passes through* B, *the circle whose chord is* E B, *and tangent* E T, *shall have the same curvature with the curve* E M H *at the point* E; *and the contact of* E M *and* E R *shall be always the closer, the less the* ∠ *is, that is contained at* B *by the curve* B K F, *and the circle of curvature* B Q E.

Let T K meet the ⊙ in R and Q; then $RT \times TQ = ET^2 = MT \times TK$ (by hypothesis) ∴ R T : M T :: T K : T Q. Suppose first that B K, the part of the curve B K F that is next to the point B adjoining to it, falls without the ⊙ B Q, and suppose T K, by moving parallel to itself, to approach to E B till it coincide with it; and while the point K describes K B, T K being greater than T Q, R T must be greater than M T, and the arc E M of the curve must pass without the ⊙ E R, betwixt it and the tangent E T: and since any ⊙ described through E, upon a chord *less* than E B touching E T, falls within the ⊙ E R B, it is manifest that no such ⊙ can pass betwixt the curve E M and ⊙ E R B. Nor can any ⊙ E *r b* described upon a chord E *b greater* than E B touching E T pass between E R and E M; for let T K meet this ⊙ in *r* and *q*, then $rT \times Tq = ET^2 = MT \times TK$; ∴ M T : *r* T :: T *q* : T K, and since F K B (by hypothesis) passes through B so that the part of it, that is next adjoining B, must be within the arc *b q* of the ⊙ *b q* E, it follows that while K describes this part of F K B, T *q* must be greater than T K; and ∴ M T greater than *r* T. Therefore the arc E *r* of the ⊙ E *r b* is without the curve E M, and passes betwixt it and the tangent E T. Hence no ⊙ whatever can pass betwixt E M and E R; and consequently the ⊙ E R B has the same curvature with E M at E. Suppose now that the part of the curve B K F, that is next adjoining to B, falls within B Q *(Fig.* 31); then while K describes this part of the curve F K B, T K being less than T Q, R T must be less than M T, and the arc E M must fall within E R; and since any ⊙ described through E, upon a chord greater than E B, falls without the ⊙ E R, it is manifest that no such ⊙ can pass betwixt E R and E M. Nor can any ⊙ E *r b* described upon a chord E *b* less than E B touching E T, pass between E R and E M; for let T K meet this ⊙ in *r* and *q*, and M T being : *r* T :: T *q* : T K, and T *q* being less than T K while K describes K B, M T must be less than *r* T; and consequently the arc E *r* must fall within E M. Therefore, in either case, all the circles that can be described through E fall without both E R and E M, or within them both; and no ⊙ whatever can pass between them when the rectangle M T × T K is always = E T^2, and the curve in which K is always found passes through B; *i. e.* the ⊙ E R B and the curve E M have the same curvature at E, which was the first part of the proposition.

Let E *m*, *(Fig.* 32*)* any other curve touching E T in E, and *f k* B, another curve passing through B, meet T K in *m* and *k*; and let the rectangle *m* T × T *k* be likewise always = E T^2; then the curvature of E *m* at E shall be the same as that of the ⊙ E R B, or that of the curve E M, by what has been demonstrated. Because *m* T × T *k*, M T × T K, R T × T Q are equal to each other, T *m* : T M :: T K : T *k* and T *m* : T R :: T Q : T *k*. Therefore if the arc B *k* pass between B K and B Q, the curve E *m* must pass between E M and E R so that E *m* must have a closer contact with this ⊙, than E M has with it: and the less the ∠ is, that is formed by the curve F K B and the ⊙ of curvature E Q B at B, the closer is the contact at E of the curve E M H, and the ⊙ of curvature E Q B. Thus the curve B K F, by its intersection with E B, determines the curvature of E M; and by the ∠ in which it cuts the ⊙ of curvature it determines the degree of contact of E M and that ⊙; the ∠ B E T and the right line E T being given.

Cor. 1. Since $MT \times TK = ET^2$, $TK = \frac{ET^2}{MT}$. Now let M move up to E and coincide with it, then will T K ultimately coincide with, and be equal to, E B; ∴ in all cases, whatever be the curve, the chord of the ⊙ of curvature = the ultimate value of $\frac{ET^2}{MT}$, or = the ultimate value of $\frac{EM^2}{MT}$.

Cor. 2. It appears from the demonstration, that according as the arc B K falls without or within the arc B Q, the arc E M falls without or within the ⊙ E R B; that when the curve F K B cuts the ⊙ E R B in B, the curve H M E cuts the ⊙ of curvature in E; that when the curve F K B is on the same side of the ⊙ B Q E on both sides of B, the curve H M E, continued on both sides of E, is on the same side of the ⊙ of curvature: and that the contact of the curve E M H and the ⊙ of curvature is closest when the curve B K touches the arcs B Q in B, the ∠ B E T being given; but is farthest from this, or is most open, when B K touches the right line E B in B.

Cor. 3. There may be indefinite degrees of more and more intimate contact between a ⊙ E R B and a curve E M H. The 1st degree is when the same right line touches them both in the same point; and a contact of this sort may take place betwixt any ⊙, and any arc of any curve. The 2d is when the curve E M H and ⊙ E R B have the same curvature, and the tangents of the curve B K F and ⊙ B Q E intersect each other at B in any assignable angle. The contact of the curve E M and ⊙ of curvature E R at E is of the 3d degree or order, and their osculation is of the 2d when the curve B K F touches the ⊙ B Q E at B, but so as not to have the same curvature with it. The contact is of the 4th degree or order, and their osculation of the 3d, when the curve B K F has the same curvature with the ⊙ B Q E at B, but so as that their contact is only of the 2d degree: and this gradation of more and more intimate contact, or of approximation towards coincidence, may be continued indefinitely; the contact of E M and E R at E being always of an order two degrees closer than that of B K and B Q at B. There is also an indefinite variety comprehended under each order. Thus when E M and E R have the same curvature, the ∠ formed by the tangents of B K and B Q admits of indefinite variety, and the contact of E M and E R is the closer the less that ∠ is. And when that ∠ is of the *same* magnitude, the contact of E M and E R is the closer the greater the ⊙ of curvature is; for since T R : T M :: T K : T Q, div°. R M (which subtends the ∠ of contact M E R) : T R :: K Q : T K, and ∴ R M : K Q :: R T × T Q (E T²) : K T × T Q; ∴ when E T is given, R M varies as $\frac{KQ}{KT \times TQ}$, and when K Q (or ∠ K B Q) is given, R M is less, in proportion as the rectangle K T × T Q, which ultimately = chord of curvature², is greater. When B K touches the ⊙ B Q at B, it may touch it on the same or on different sides of their common tangent; and the ∠ of contact K B Q may admit of the same variety with the ∠ of contact M E R in the former case. But there is seldom occasion for considering these higher degrees of more intimate contact of the curve E M H, and ⊙ of curvature E R B.

Cor. 4. The curvature is uniform in the ⊙ only.

When the curvature of E M H encreases from E towards H, and consequently corresponds to that of a ⊙ gradually less and less, the arc E M falls within E R, and B K is within B Q. When the curvature of E M decreases from E towards H, and consequently corresponds to that of a ⊙ that is gradually greater and greater, the arc E M falls without E R, and B K is without B Q. According as the curvature of E M varies more or less, it is more or less unlike to the uniform curvature of a ⊙, the arc of the curve E M H separates more or less from the arc of the ⊙ of curvature E R B, and the ∠ contained by the tangents of B K F and B Q E at B is greater or less. And thus the *quality* of curvature, (as it is called by Sir I. Newton) depends on the ∠ contained by the tangents of B K and B Q at B.

Cor. 5. Let the curve E M H for example, (*Fig.* 33) be a parabola, E B a diameter, E T the tangent at E, then because parameter $\times$ T M $=$ E T^2 $=$ M T $\times$ T K, T K is always $=$ the parameter, ∴ in this case B K is a straight line parallel to the tangent E T, which intersects E B in B, so that E B is $=$ that parameter. Therefore if upon the diameter of a parabola, a right line E B be taken from E the vertex of this diameter $=$ to its parameter, a ⊙ E R B, described upon this right line as its chord, that touches the parabola at E, shall be the ⊙ of curvature. And because the right line B K cuts the ⊙ B Q E in B, unless when E is the vertex of the figure, the parabola cuts the ⊙ of curvature (that case excepted); and passes within the ⊙ of curvature when it is produced towards the vertex, but without it when produced the contrary way.

Cor. 6. When E B does not meet with the curve F K, (*Fig.* 34) but is its asymptote; any ⊙ being described touching E T in E, a greater ⊙ shall always pass between it and the curve E M; and the greater this ⊙ is, the closer shall its contact be with the curve E M. For since the curve F K produced passes without any ⊙ E Q B, how great soever, that can be described through E, E M must always pass betwixt E R and the tangent E T. This is the case in which the curvature is said to be infinitely small, (being less than that of any ⊙) or the ray of curvature infinitely great. Of this we have an example in the vertex of the cubical parabola; for in that case $ET^3 = TM \times a^2$ (where a^2 is a given square) ∴ $\frac{ET^3}{TM} = a^2$, but $\frac{ET^2}{TM} = TK$, ∴ $\frac{ET^3}{TM} = TK \times ET$, hence $ET \times TK =$ the given square a^2; ∴ the curve F K is the common hyperbola, whose asymptotes are E B and E T. The curvature is of the same kind at the vertex of any parabola, wherein T M is as any power of E T, whose exponent exceeds 2; for F K, in all those cases, is an hyperbola, of which E B is an asymptote.

Cor. 7. When the curve F K (*Fig.* 35) passes through E, no ⊙ can be described through E so small, but a less ⊙ shall pass between it and the curve E M, and the less this ⊙ is, the closer shall its contact with E M be. For since the curve F K passes within any ⊙ that can be described through E on the same side of E T, the arc E M is always within E R. In this case, because the curvature surpasses that of any ⊙, it is said to be infinitely great, or the ray of curvature to be infinitely small. Of this we have an example at the vertex or cuspid of the semicubical parabola; for in that case $ET^3 = MT^2 \times a$, (where a is a given line) ∴ $\frac{ET^3}{MT^2} = a$, and $\frac{ET^4}{MT^2} = a \times ET$; but $\frac{ET^2}{MT} = TK$, ∴ $\frac{ET^4}{MT^2} = TK^2$, hence $a \times ET = TK^2$; ∴ F K E is the common parabola, whose *latus rectum* $= a$, and which touches E B in E.

Lemma 11.

55. *Case* 1. It follows, from Cor. 1. Art. 54, that if A G, drawn perpendicular to A D, and B G, perpendicular to A B, intersect each other in G, the limit to A G is the chord of curvature A I. For by similar △s G A : A B :: A B : B D, ∴ G A = $\frac{A B^2}{B D}$, and consequently their limits are equal: but the limit of $\frac{A B^2}{B D}$ is the chord of curvature, (by Cor. 1.) ∴ also the ultimate value of A G is the chord of curvature, or A G ultimately = A I. The proof of the Lemma is ∴ evident.

Case 2. Let B D and *b d* (*Fig.* 36) be equally inclined to A D at any given ∠; draw B E, *b e* perpendicular to A D, then by similar △s B D : *b d* :: B E : *b e*; *i. e.* in the given R°. of A B^2 : A b^2 by the first case.

Case 3. Let the ∠s at D and *d* (*Fig.* 36) be not equal, *i. e.* let B D, *b d* converge to some point O, at a finite distance. Draw B E, *b e* perpendicular to A D, then when A B, A *b* are diminished without limit. their difference B *b* will be diminished without limit; ∴ the ∠ B O *b* will be diminished without limit; but ∠ B O *b* = ∠ A *d* O — ∠ A D O; ∴ the ∠ A *d* O = ∠ A D O ultimately, and consequently B D E, *b d e* are ultimately similar, and B D : *b d* :: B E : *b e*, i. e. in the ultimate R°. of A B^2 : A b^2.

Lemma 11.—*Cor.* 2.

56. Let the sagittæ E F, *e f* (*Fig.* 37) bisecting the chords A B, A *b*, meet in H; join A H and produce it to K, making A H = H K; join K B, K *b* and produce them to D, *d*. By construction A H : A K :: A F : A B, ∴ H F, K B or F L, B D are parallel. When B moves up to A, the ultimate R°.

of E L : B D is that of $AE^2 : AB^2$ (by Lem.) or that of $AF^2 : AB^2$ or that of 1 : 4 (for A F, A E are ultimately equal). But B D : F L :: A B : A F :: 4 : 2, ∴ E L : F L ultimately :: 1 : 2, consequently F E, E L are ultimately equal, and ∴ E F is ultimately to B D :: 1 : 4. In like manner *e f* is ultimately to *b d* :: 1 : 4; ∴ E F : B D :: *e f* : *b d* ultimately, and E F : *e f* :: B D : *b d* ultimately; but B D, *b d* converge to a given point K, ∴ (Lem. Case 3), the points B, *b* meeting in A, B D, *b d* and consequently E F, *e f* are ultimately as the squares of A B, A *b*.

Lemma 11.—*Cor.* 5.

57. By Cor. 1. A C : A *c* :: $CB^2 : cb^2$ ultimately, (*Fig.* 38) which is the property of the parabola; ∴ the curve A B, whatever be its nature, provided it be of finite curvature (see Schol.) may ultimately be considered as a parabola; ∴ the curvilinear area A C B $= \frac{2}{3}$ C D ultimately, and consequently the curvilinear area A D B $= \frac{1}{3}$ C D ultimately $= \frac{2}{3}$ of the △ ADB ultimately, and consequently the remainder, the segment A B, $= \frac{1}{3}$ △ A D B ultimately; but △ A D B varies as AD^3 or AB^3 ultimately (Cor. 4); ∴ also the curvilinear area A D B and segment A B vary as AD^3 or AB^3 ultimately.

SCHOLIUM.

Introductory Articles to Scholium.

58. Prop. 1. *Let there be two curves of any kind* (Fig. 39) A B, A b, *and suppose the ∠ of contact* B A D *in the* 1*st case to be indefinitely greater than the ∠ of contact* b A D *in the other; then shall the curvature of* A B *be indefinitely greater than that of* A b; *and conversely.*

Let A I, A *i* be the diameters of curvature of A B and A *b* respectively; then $A I = \frac{A D^2}{B D}$ ultimately, and $A i = \frac{A D^2}{b D}$ ultimately, ∴ A I : A *i* in the ultimate R°. of $\frac{A D^2}{B D} : \frac{A D^2}{b D}$, *i. e.* in the ultimate R°. of *b* D : B D. Now the ∠ B A D is indefinitely greater than *b* A D by hypothesis, but the ultimate R°. of B D : *b* D is the same with that of those ∠s., for they ultimately measure them; ∴ ultimately B D is indefinitely greater than *b* D, ∴ A *i* is also indefinitely greater than A I; but the curvature varies as $\frac{1}{D^r. \text{ of curv}^e.}$; ∴ the curvature of A B is indefinitely greater than that of A *b*.

Next let the curvature of A B be indefinitely greater than that of A *b*, then shall the ∠ B A D be indefinitely greater than the ∠ *b* A D; for as before A I : A *i* in the ultimate R°. of *b* D : B D, and A *i* is indefinitely greater than A I by hypothesis, ∴ B D is ultimately indefinitely greater than *b* D, and consequently the ∠ B A D indefinitely greater than the ∠ *b* A D.

59. Prop. 2. *Let there be two curves* A B, A b, *and let the* ∠ *of contact* B A D *bear a finite Ratio to the* ∠ *of contact* b A D; *then if the curvature of* A B *be finite, the curvature of* A b *will also be finite; and conversely if the curvature of* A B, A b *be both finite, the* ∠s *of contact* B A D, b A D *will be to each other in a finite Ratio.*

For, as before, A I : A *i* in the ultimate R°. of *b* D : B D; but *b* D : B D ultimately in a finite R°. by hypothesis, ∴ A I : A *i* in a finite R°., but A I is finite, ∴ also A *i* is, and consequently curvature o A *b* is, finite.

Again A I : A i in the ultimate R°. of b D : B D; but the R°. of A I : A i is finite by hypothesis ∴ the ultimate R°. of b D : B D, and consequently that of the ∠ˢ. of contact, is finite.

Cor. 1. Let A B be any ⊙, then since the curvature of a ⊙ is always finite, it is manifest that the curvature of all curves, whose ∠ˢ of contact bear a finite R°. to that of this ⊙; or, which is the same thing, the subtenses of whose ∠ˢ of contact bear ultimately a finite R°. to that of this ⊙, will be finite; and if the limiting R°. of the subtenses of the ∠ of contact of the curve and ⊙ be not only finite, but also a R°. of equality, then the curve and ⊙ have the *same* curvature at the point of contact.

Cor. 2. Since A I : A i :: $\frac{1}{BD} : \frac{1}{bD}$, the curvatures of two curves are to each other as the ∠ˢ of contact, or as the *ultimate* subtenses of these angles.

Scholium.

60. In the above Lemma, the ∠ of contact is supposed to bear a finite R°. to that of a ⊙, *i. e.* the curvature is supposed to be neither indefinitely great, nor indefinitely small (Cor. 1. Art. 59.) This is manifest from the Lemma itself, which was proved on the supposition that the diameters A G, A g had a limit, viz. A I; *i. e.* that the curve had a ⊙ of curvature. To shew, however, this in another point of view, it may be worth while to prove (1) That, conversely to the Lemma, if B D vary as A D² ultimately, the curvature of A B is finite. (2) That if B D ultimately vary in any other R°. greater or less than that of A D², the curvature is not finite, but infinitely small or infinitely great. (3) That there may be curves, whose curvatures are indefinitely great or indefinitely small, and again curves, whose curvatures are indefinitely greater or indefinitely smaller than that of those others, and so without end; and thus

that the ∠ of contact B A D may be divided into a series of ∠s, each of which is indefinitely greater or indefinitely smaller than the one which is adjacent to it, and that this division may be continued *sine limite.*

(1) Let A E V (*Fig.* 40) be *any* ⊙, and A B the curve, then since B D ultimately varies as AD^2 (by hypothesis) B D ultimately $= \frac{AD^2}{a}$ (where *a* is a proper constant quantity), but E D ultimately $= \frac{AD^2}{AV}$, ∴ the ultimate R°. of B D : E D = that of $\frac{AD^2}{a} : \frac{AD^2}{AV}$ = that of A V : *a*, which last R°. is always finite, whatever be the value of A V provided it be finite, and ∴ the ultimate R°. of B D : E D is finite, and ∴ the curvature of A B is finite, (by Cor. 1. Art. 59.)

(2) Let B D (*Fig.* 41) ultimately vary in any R°. greater than that of AD^2, for instance AD^3, then B D ultimately $= \frac{AD^3}{a^2}$ (where *a* is a proper constant quantity), also as before E D ultimately $= \frac{AD^2}{AV}$, ∴ B D : E D :: $\frac{AD^3}{a^2} : \frac{AD^2}{AV}$ ultimately, *i. e.* :: A D : $\frac{a^2}{AV}$ ultimately; but in the ultimate state A D is indefinitely less than $\frac{a^2}{AV}$, whatever be the value of A V provided it be finite, ∴ B D is ultimately indefinitely less than E D; and ∴ the curvature of A B is indefinitely small, (by Art. 58) *i. e.* no ⊙ however great, can pass between the curve A B and tangent A D, as appeared also from Cor. 6. Art. 54. And the same may be shewn when B D ultimately varies as AD^4, AD^5, AD^6......AD^n, where *n* (pro-

vided it be greater than 2) may be any N°. whatever, whole or fractional.

Next let B D (*Fig.* 40) ultimately vary in any R°. less than that of AD^2, for instance $AD^{\frac{3}{2}}$, then BD ultimately $= \frac{AD^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, $\therefore BD : ED :: \frac{AD^{\frac{3}{2}}}{a^{\frac{1}{2}}} : \frac{AD^2}{AV}$ ultimately, *i. e.* $:: \frac{AV}{a^{\frac{1}{2}}} : AD^{\frac{1}{2}}$ ultimately; but in the ultimate state, $AD^{\frac{1}{2}}$ is indefinitely less than $\frac{AV}{a^{\frac{1}{2}}}$, whatever be the value of A V provided it be finite; ∴ E D is ultimately indefinitely less than B D, and ∴ the curvature of A B is indefinitely great; *i. e.* there can be no ⊙, however small, which does not pass without the curve (by Art. 58); as appeared also from Cor. 7, Art. 54. And the same may be shewn when B D ultimately varies as $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$ .....AD^n, where n (provided it be less than 2) may be any fractional N°. whatever.

(3) *(j)* Let BD (*Fig.* 42) ultimately vary as AD^2, then, as we have seen above, the curvature of A B is finite.

(jj) Let A P be another curve, such that D P ultimately varies as AD^3, then will D P ultimately $= \frac{AD^3}{a^2}$; also B D ultimately $= \frac{AD^2}{b}$ (where a and b are proper constant quantities); $\therefore BD : PD :: \frac{AD^2}{b} : \frac{AD^3}{a^2}$ ultimately, $:: \frac{a^2}{b} : AD$ ultimately;

but A D is ultimately indefinitely less than $\frac{a^2}{b}$, ∴ P D is ultimately indefinitely less than BD, or curvature of A P is indefinitely small, as we have before seen.

(jjj) Again, let A C be another curve, such that

$C D$ ultimately varies as $A D^4 =$ ultimately $\frac{A D^4}{m^3}$,

$\therefore P D : C D :: \frac{A D^3}{a^2} : \frac{A D^4}{m^3} :: \frac{m^3}{a^2} : A D$ ultimately, but $A D$ is ultimately indefinitely less than $\frac{m^3}{a^2}$, $\therefore C D$ is ultimately indefinitely less than $P D$, or curvature of $A C$ is indefinitely less than that of $A P$, which is indefinitely small. And in the very same manner, if the subtense ultimately varies as $A D^5$, $A D^6$, &c., we shall have a series of $\angle^s$ of contact going on in infinitum; each of which is indefinitely less than the preceding. Also between any two of these $\angle^s$ there may be inserted a series of intermediate $\angle^s$ going on in infinitum, any one of which is indefinitely less than the preceding. For instance, between $A D^2$ and $A D^3$ there may be inserted the series $A D^{2\frac{3}{8}}$, $A D^{2\frac{1}{5}}$, $A D^{\frac{9}{4}}$, $A D^{\frac{5}{2}}$, $A D^{\frac{8}{3}}$, $A D^{2\frac{1}{4}}$, $A D^{2\frac{4}{5}}$, &c. &c. And again, between any two $\angle^s$ of this series, there may be inserted a new series of intermediate $\angle^s$, differing from each other by infinite intervals, and so on without limit.

Next (*j*) let $A E$ be a curve, such that $E D$ ultimately varies as $A D^{\frac{3}{2}}$, $=$ ultimately $\frac{A D^{\frac{3}{2}}}{a^{\frac{1}{2}}}$; then

$E D : B D :: \frac{A D^{\frac{3}{2}}}{a^{\frac{1}{2}}} : \frac{A D^2}{b} :: \frac{b}{a^{\frac{1}{2}}} : A D^{\frac{1}{2}}$ ultimately, but $A D^{\frac{1}{2}}$ is ultimately indefinitely less than $\frac{b}{a^{\frac{1}{2}}}$, $\therefore$ ultimately $E D$ is indefinitely greater than $B D$, or curvature of $A E$ is indefinitely great, as has been before shewn.

(*jj*) Again, let $A F$ be another curve, such that $F D$ ultimately varies as $A D^{\frac{4}{3}} = \frac{A D^{\frac{4}{3}}}{m^{\frac{1}{3}}}$ ultimately, then F D : E D :: $\frac{A\,D^{\frac{4}{3}}}{m^{\frac{1}{3}}}$: $\frac{A\,D^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ :: $\frac{a^{\frac{1}{2}}}{m^{\frac{1}{3}}}$: A $D^{\frac{1}{6}}$ ultimately, but A $D^{\frac{1}{6}}$ is ultimately indefinitely less than $\frac{a^{\frac{1}{2}}}{m^{\frac{1}{3}}}$; ∴ ultimately F D is indefinitely greater than E D, or curvature of A F is indefinitely greater than that of A E, which is indefinitely great. And in the very same manner if the subtense varies as A $D^{\frac{5}{4}}$, A $D^{\frac{6}{5}}$, A $D^{\frac{7}{6}}$.....A D^{n}, (n being any fractional N°. whatever less than 2) we shall have a series of ∠s of contact running on in infinitum, each of which is indefinitely greater than the one which precedes it. "Neque novit natura limitem."

NOTES TO SECTION II.

INTRODUCTORY ARTICLES TO SECTION II.

61. *Defn.* Whatever tends constantly to solicit or impel a body towards a fixed point or centre, is called a *centripetal force.*

The centripetal force, which is found to exist in the sun and planets, is, by way of distinction, called *gravity*, or the *force of gravity.*

62. The word gravity is used in three different senses, or rather it is spoken of as being greater or less in reference to three different measures. As (1) we may say for instance that the gravity of the earth, at the distance of one mile from its surface, is greater than the gravity of the earth, at the distance of 1000 miles from its surface. By this proposition we mean that the *velocity* uniformly generated in a given time, in a body at one mile's distance from the earth's surface, is greater than the velocity uniformly generated in the same given time, at the distance of 1000 miles from it. The word, when used in this sense, is called the accelerating force of gravity; and, in general, when we speak of the force of gravity at different distances from the same attracting body, the accelerating force of gravity is always understood. Hence the following definition. 'When the velocity uniformly

produced in a given time is the measure by which gravity is said to be greater or less; then it is called the *accelerating force of gravity.*'

This accelerating force of gravity is in all cases found to be invariably the same at equal distances from the centre of the same attracting body, and to vary according to some regular law of the distance from that centre; and hence it is, that the variation of this force is usually expressed in terms of the distance from the centre of the attracting body; for instance, when it is said that gravity varies as the *nth* power of the distance, the expression denotes that the accelerating force of gravity (measured by the velocity uniformly generated in a given time) increases or decreases as the n^{th} power of the distance from the centre increases or decreases; and $F \propto D^n$ is called the law of the accelerating force.

(2) Again we may say that the gravity exerted upon a cubic inch of gold is greater than that upon a cubic inch of cork. Here we no longer refer to the same measure as before, but mean by the Prop. that the *quantity of motion*, uniformly generated in a given time in the gold, is greater than that uniformly generated in the same time in the cork, when placed at an *equal* distance from the attracting body's centre; or in other words, that the weight of the gold is greater than the weight of the cork. The word, when used in this second sense, is called the motive force of gravity, and as, when speaking of gravity at different distances from the centre of the same attracting body, we mean the accelerating force of gravity; so, when speaking of the gravity exerted upon different bodies at the same distance, the motive force of gravity is to be understood. Hence the following definition. 'When gravity is considered as greater or less in proportion to the quantity of motion it uniformly produces in a given time, then it is called the *motive force of gravity.*'

The only difference then betwixt the accelerating and motive force of gravity is this, that inasmuch as gravity produces both velocity and momentum, we call it one or the other, according, as for the sake of convenience, the velocity or momentum is taken to be the measure of it.

(3) Lastly, we frequently speak of the gravity of different attracting bodies, as when we say that the gravity of the earth is greater than the gravity of the moon. By this Prop. it is meant that the accelerating force of the earth, at a given distance from its centre, is greater than the accelerating force of the moon at the same given distance from *its* centre; *i. e.* that the velocity, uniformly generated in a body in a given time, and at a given distance from the earth's centre, is greater than the velocity uniformly generated in the same time, and at the same distance from the moon's centre. The word, when used in this last sense, is called the absolute force of gravity; and when the gravity of different attracting bodies is spoken of, the absolute force of gravity (measured in the manner above described) is always understood. Hence the following definition. 'When gravity is considered as greater or less, in reference to the efficacy of the cause which produces it, then it is called the *absolute force of gravity.*'

63. *The accelerating forces, acting upon bodies, at different distances from different centres of force, are as the absolute forces, and the law of the force jointly; i. e. if* Φ *and* φ *represent the absolute forces,* D *and* d *the two distances, and the law of the force be the direct* n^{th} *power of the distance;* $F : f :: \Phi \times D^n : \varphi \times d^n$.

For if the distances of the two bodies from their respective centres be the same, the accelerating forces are the same with the absolute forces, *i. e.* if $D = d$; $F : f :: \Phi : \varphi$; and if the absolute forces be the same, *i. e.* if $\Phi = \varphi$; $F : f :: D^n : d^n$; $\therefore$ when both the absolute forces and distances are different, $F : f :: \Phi \times D^n : \varphi \times d^n$.

Cor. If f, φ, and $d = 1$; F will be represented by $\Phi \times D^n$, or by the absolute force and the law of the force.

64. *The motive forces* M *and* m, *acting upon different bodies, at different distances from different centres of force; in other words, the weights or tendencies of different bodies towards different centres are in a joint Ratio of the quantities of matter in the bodies attracted, the absolute forces of the attracting bodies, and the law of the force.*

For by last Art. $F : f :: \Phi \times D^n : \varphi \times d^n$.
but in all cases $M : m :: Q \times F : q \times f$
$\therefore M : m :: Q \times \Phi \times D^n : q \times \varphi \times d^n$.

Cor. If m, q, φ and d be all taken $= 1$; M will be represented by $Q \times \Phi \times D^n$.

PROPOSITION I.

Note to Prop. 1.

65. Since (*Fig.* 43) the $\triangle$s S A B, S B c, S c d, &c. are always equal to each other, and to the $\triangle$s S A B, S B C, S C D, &c. the whole S A B d is equal to the whole polygon S A B C D, and their limits will be equal; but the limiting position of A B d is that of a tangent at A, and the limit of the polygon S A B C D is the curvilinear area S A B C D; if $\therefore$ A d (*Fig.* 44) be the space described in the tangent with the velocity at A continued uniform, in the time that the body describes A D with a variable velocity, the area S A d will be equal to the area S A D.

Note to Prop. 1.—*Cor.* 1.

66. If the areas described in a given time are not equal, *i. e.* if bodies move in different orbits, the bases

of the $\triangle^s$, which in all cases represent the velocities, will be as those $\triangle^s$ directly, and the perpendiculars upon the bases inversely, *i. e.* by taking the limiting R^{os}., the velocities of bodies revolving in different orbits are at any points of the orbits universally as the areas described in a given time directly, and the perpendiculars upon the tangents to those points inversely. Hence, if the time be denominated 1, V = A B, but $A B = \frac{2 S A B}{\text{perp.}}$, $\therefore V = \frac{2a}{p}$ where a = area described in a given time, and p = perpendicular.

Prop. 1.—*Cor.* 2.

67. Suppose first the body to describe uniformly the chords themselves A B, B C; join A V, then since C V is = and parallel to B c, it is also = and parallel to A B; ∴ B V, which passes through the the centre S, is the diagonal of the parallelogram A B C V; now since the position of B V will not be altered by the magnitudes of A B, and B C, let them be diminished in infinitum, then will they ultimately coincide with the chords of two arcs successively described in equal times (when those arcs are diminished in infinitum), and B V, which always passes through the centre, will ultimately coincide with the diagonal of the parallelogram formed by those chords.

Prop. 1.—*Cor.* 3.

68. If the body actually moved over A B, B C; D E, E F, &c. and the force acted impulsively, the force at B would be to the force at E accurately as B V to E Z, they being the uniform effects of the force at those points; but if the force act incessantly, and consequently A B, B C; D E, E F be diminished in infinitum, the force at B will be to the force at E in the ultimate R^o. of B V : E Z, *i. e.* as in the last Cor. in the ultimate R^o. of the diagonals of the parallelograms formed by the chords of arcs successively described in equal times.

Prop. 1.—*Cor.* 4.

69. Draw the diagonals C A, D F, which will bisect B V, E Z in *m* and *n*, then (Cor. 3) Fce at B : Fce at E in the ultimate R°. of B V : E Z or in the ultimate R°. of B *m* : E *n*; but the ultimate magnitudes and positions of B *m*, E *n* are those of the sagittæ of two arcs A B C, D E F described in equal times, which converge to the centre S, and bisect the chords A C, D F when these arcs are diminished in infinitum.

Prop. 1.—*Cor.* 5.

70. The parabolic arc described by a body falling obliquely at the earth's surface may be deduced in the same manner from the polygonal motion, only in this case the sagittæ will be equal and parallel to each other; these sagittæ may, as in the former case, be proved to be measures of the force, *i. e.* of the force of gravity at the earth's surface; hence the force of a body moving in any curve will be to the force of gravity in the ultimate R°. of the sagittæ of the arcs described in equal times in the two cases. Now the sagitta of the parabolic arc described in a very small time, as one second, is known by experiment in feet; if ∴ we can find the sagitta of the arc of any other curve described in the same small time in feet, we can make a direct comparison between the centripetal force in the curve and that of gravity.

PROPOSITION II.

71. Let us first suppose that the body describes the polygon A B C D F formed by the chords of this curve, and that it is deflected only at the ∠s B, C, D, &c.; then since B *c* = A B, the body, if not act-

ed upon by any force, would at the end of the second portion of time be found in *c*, having described B*c*; but it is really found in C at that time, having described B C; C*c* ∴ which completes the △ B C *c* must represent the quantity and direction of the force acting at B, since it is the motion which, when combined with B*c*, produces B C the real motion; *i. e.* the force at B must act in a direction parallel to C*c*; but since S B C (= S A B) = S B*c*, C*c* and S B are parallel, ∴ force at B acts in direction B S; and it may be shewn in like manner, that the force at C, D, E, &c. is directed to the same point S. Now let the sides of this polygon be diminished and their N°. increased ad infinitum, in which case the force acts incessantly, and the body describes a curve line; the demonstration still remains the same, since it did not at all depend upon the magnitudes of A B, B C, &c.

Prop. 2.—Cor. 1.

72. Let S A B, S B*c* (*Fig.* 45) be two equal △s as in the Prop.; draw *c* D parallel to B S, then if S B C be greater than S A B, *i. e.* if the description of the areas be accelerated, the vertex of S B C must fall without *c* D, ∴ if *c* C be joined, and B F be drawn parallel to it, the centre of force S will be in B F, and ∴ must have moved up from S into that line, or it has declined towards that quarter towards which the body is going; and in the very same manner when the description of the areas is retarded, the vertex of the △ will fall within *c* D, or the force will decline to the other side of S, *i. e.* in antecedentia.

OBSERVATIONS ON THE TWO LAST PROPOSITIONS.

On Polygonal and Curvilinear Motions.

73. Let A B C D (*Fig.* 46) be a polygon described

by a body round S, and suppose the straight lines A B, B C, C D, &c. to be described in the same indefinitely small time T. Now of this motion of a body in a polygon, it may be observed, (1) That the force acts only by impulses, which succeed each other after equal intervals, viz. when the body is at the points B, C, D, &c., and consequently that the uniform motion of the body in any side of the polygon, as B C, is compounded of two uniform motions; one which would carry it in the original direction which it had at B, viz. through B E (= A B) in the given time T; and the other, which would uniformly carry it through E C, parallel to B S, in the same time T. (2) That this uniform velocity towards the centre is generated the moment the body arrives at B, by the instantaneous impulse of the force, and is just equal to that which a body would acquire by falling from rest in the given time T, by the uniform action of the same force.

Now let us, in the next place, suppose the body to describe the *curve* A B C D, and to be found in the points A, B, C, D, &c. in the same instants of time that the body in the polygon was. Then the body, when at B, will no longer have the direction B E as in the polygon, but the direction B G, which is a tangent to the curve at B; since then, if the force had not acted, the body would have been found in B G, but is really found in C; it is evident that C G must be the space through which the force has drawn the body in the given time T; which line C G, since the force in the indefinitely small time T will not change its direction, must coincide in position with C E. Now it will be shewn, Art. 80, that C A ultimately = 2 C L, and that B G is parallel to C L; ∴ C A ultimately = 2 B G, and ∴ E C ultimately = 2 G C; that is, the deflection in the polygonal motion is ultimately just double the contemporaneous deflection in the curvilinear.

This difference in the deflection is what constitutes the chief distinction betwixt a polygonal and curvilinear motion; and a very little consideration will shew that it is just what ought to take place, from the difference of the hypotheses in the two cases. For since curvilinear motion is a case of *continued* deflection, the velocity towards the centre, in any one indefinitely small portion of time, is a *variable* velocity *beginning from nothing;* whereas in the polygonal motion it is the velocity so acquired continued *uniform* for the same time; consequently, since the force for the indefinitely small time T will be constant, the space described in the former case ought to be only half what is described in the latter. Hence it is perfectly legitimate to reason from a polygonal to a curvilinear motion, and the only difference between them is this: that as in the curvilinear motion the force acts incessantly, so, to make up for this, there is a proper corresponding diminution in the space through which it has to draw the revolving body.

Cor. 1. Hence the force is measured, both in the polygonal and curvilinear motion, by the same quantity, viz. by the ultimate value of E C or 2 G C; for in the former case E C being the space uniformly described by the action of the force in the given time in which B E is described, is a proper measure of the intensity of that force; and, in the latter case, since G C is a space freely described from rest in the same given time, 2 G C will be a measure of the fluxion of the velocity uniformly generated in that time, or a measure of the force.

Cor. 2. Let S B $= y$, then G C being the deflection of the curve from the tangent ultimately $= \frac{1}{2} \ddot{y}$, $\therefore$ force in curve (varies as 2 G C) varies as $\ddot{y}$.

Cor. 3. Though the force in the curve is properly measured by *twice* the subtense of the arc described in an indefinitely small given time; yet when the forces to be compared together, are all computed in the same way, it matters not whether we take the subtenses, (as Newton generally does, see Prop. 1.

Cor. 4) or their doubles, as the measures of them; the R°. being the same in both cases: Nevertheless, when the forces so found are to be compared with others derived from a fluxional calculus, (which has always a reference to the polygon) it is absolutely necessary to take the double subtense for the measure of the force.

PROPOSITION IV.

74. Since the bodies (*Fig.* 47) move equably in the ⊙s, equal areas will in each case be described in equal times; consequently equal sectors or areas will be described round the centres S, *s* in equal times; ∴ the centripetal forces tend to the centres of the ⊙s. Again let B A E, *b a e*, be two arcs described in the same indefinitely small time, then A C, *a c*, which bisect the chords and tend to the centres of the ⊙s will be the sagittæ of these indefinitely small arcs; ∴ Fce at A : Fce at *a* in the ultimate R°. of A C : *a c*, or of

$$\frac{\overline{\text{chord } AB}^2}{AG} : \frac{\overline{\text{ch. } ab}^2}{ag} \text{ or of } \frac{\overline{\text{arc } AB}^2}{AG} : \frac{\overline{\text{arc } ab}^2}{ag}$$

(Lem. 7). Now let A F, *a f*, be *any* two arcs described in equal times; then, since the motions are uniform, A B : *a b* :: A F : *a f*, ∴ F at A : F at *a* ::

$$\frac{\overline{\text{arc } AB}^2}{AG} : \frac{\overline{\text{arc } ab}^2}{ag} :: \frac{\overline{\text{arc } AF}^2}{AS} : \frac{\overline{\text{arc } af}^2}{as}$$

75. If the absolute forces be different, the expression for the force is the same; for the accelerating force is in all cases proportional to the subtense of the arc described in an indefinitely small given time.

Prop. 4.—*Cor.* 2.

76. Let F and *f* be the centripetal forces of bodies describing different ⊙s, V and *v* their velocities, P

and p their periodic times; C and c the circumferences of the ⊙s, R and r their radii; then since in all cases of uniform motion, velocity varies as space directly and time inversely, $V : v :: \frac{C}{P} : \frac{c}{p} ::$ (since the circumferences of ⊙s are as their radii) $\frac{R}{P} : \frac{r}{p}$, $\therefore V^2 : v^2 :: \frac{R^2}{P^2} : \frac{r^2}{p^2}$; hence $F : f :: \frac{V^2}{R} : \frac{v^2}{r} :: \frac{R}{P^2} : \frac{r}{p^2}$.

Note to Prop. 4.—Cor. 7.

77. Let φ = absolute force, and the law of the force be $\frac{1}{R^{2n-1}}$; then if bodies revolve round *different* centres, accelerating force will (Art. 63) varies as $\frac{\varphi}{R^{2n-1}}$ $\therefore$ P varies as $\frac{R^n}{\varphi^{\frac{1}{2}}}$, and V varies as $\frac{\varphi^{\frac{1}{2}}}{R^{n-1}}$.

Introductory Articles to Prop. 4.—Cor. 8.

78. Let P Q E be any curve, (*Figs.* 48 and 49) and S any point within it; take any point s, and from it draw any line $s\,p$; suppose the radius vector S P of the curve P Q E to revolve round S, and at the same time let the line $s\,p$ begin to revolve round s, with an angular velocity always equal to that of S P, and so that $s\,p$ may always be to S P in a given R°.; then will the curve, traced out by p, be similar to the curve P Q E (Art. 30). The points S, s, are called points similarly situated; and if $\angle^s$ $p\,s\,q$, $p\,s\,c$, &c. = $\angle^s$ P S Q, P S C, &c. respectively, then p, q, c, &c., and P, Q, C, &c. are called similar points; $s\,p$, $s\,q$, $s\,c$, &c., and S P, S Q, S C, &c. similar or homologous lines; $p\,q$, $p\,c$, $q\,c$, &c., and P Q, P C, Q C, &c. similar or homologous arcs: and $p\,s\,q$, $p\,s\,c$, $q\,s\,c$, &c.,

and P S Q, P S C, Q S C, &c. similar areas of the similar figures respectively.

79. From the definition of similar figures it follows, (1) That if S, s be points similarly situated, the chords of similar arcs P Q, $p\,q$, make equal ∠s with the radius vectors S P, $s\,p$; and are to each other in a given R°.. For since P S : S Q :: $p\,s$: $s\,q$, and ∠ P S Q = ∠ $p\,s\,q$, ∴ △s P S Q, $p\,s\,q$ are similar; ∴ ∠ Q P S = ∠ $q\,p\,s$, and P Q : $p\,q$ :: P S : $p\,s$ in a given R°. (2) That the tangents to similar points P, p, make equal ∠s with the radius vectors to those points; for ∠ S P Q always = ∠ $s\,p\,q$ by the first case, ∴ they are ultimately equal; but these ∠s are ultimately the ∠s between the tangents and the radii, ∴ ∠ S P R = ∠ $s\,p\,r$. (3) That similar arcs P E, $p\,e$, as also similar areas P S E, $p\,s\,e$, of similar figures are to each other in a given R°.; for let the similar arcs P E, $p\,e$, be divided into the same N°. of similar arcs P Q, Q C; $p\,q$, $q\,c$, &c., and draw the chords; then, by the first case, these chords are to one another in a given R°., viz. in the R°. of S P : $s\,p$; consequently the sums of the chords are in the same given R°.; and since this is always the case, they are also ultimately in this given R°. Hence, Cor. Lem. IV., the arc P E : the similar arc $p\,e$ in that given R°; *i. e.* similar curves, or similar arcs of similar curves, are to one another as any similar or homologous radius vectors. And in the same manner, by dividing the similar areas into similar parts, we have the areas of similar curves, or of similar parts of similar curves to one another in a given R°., viz. in the duplicate R°. of any homologous radius vectors. (4) That the similarly situated chords of curvature P V, $p\,v$ to similar points of similar figures, are as the radius vectors to those points, or as any other homologous lines in the figures. For draw the subtenses Q R, $q\,r$ of the evanescent arcs P Q, $p\,q$ parallel to S P, $s\,p$; then, by the nature of the ⊙ of curvature, P V : P Q :: P Q : Q R; and $p\,v$: $p\,q$ :: $p\,q$:

q r; but by similar △s P Q : Q R :: *pq* : *q r*, ∴ P V : *p v* :: P Q : *pq* :: S P : *sp*, or as any homologous lines in the figures.

80. Let A P Q be any arc, (*Fig.* 48) A Q the chord of that arc; S the centre of force. Draw the radius S P bisecting the chord A Q, then will P N be the sagitta of the arc A P Q at the point P where S N meets the curve; draw the tangent B R, and the subtenses Q R and A B parallel to S P, and let P V be the chord of curvature at the point P; this being premised, it follows (1) That this sagitta will ultimately bisect the arc A P Q, or that the point P is ultimately in the middle of the arc A P Q; for since Q N = N A, and that Q N ultimately = arc Q P, and A N ultimately = arc A P, ∴ arc Q P ultimately = arc A P. (2) That the chord A N Q is ultimately parallel to the tangent P R drawn to the curve at the point P; for A B is ultimately to Q R as P R^2 or P Q^2 to P B^2 or P A^2, *i. e.* in a R° of equality; they are also parallel, ∴ A Q and P R are also ultimately parallel. (3) That the evanescent subtense Q R or A B is ultimately = to the sagitta P N, which ultimately bisects the arc A P Q; for R N is ultimately a parallelogram, ∴ Q R and P N are ultimately equal.

Prop. 4.—*Cor.* 8.

81. Let A P E, *a p e* (*Figs.* 48 *and* 49) be two similar figures, having the centres of force S, *s* similarly situated in them, P and *p* similar points of the orbit, A P Q, *a p q* two arcs described in the same time, whose middle points are ultimately P and *p*, join S P, *s p*; then since P N, *p n* ultimately bisect the arcs A P Q, *a p q*, they are ultimately the sagittæ of those arcs (Art. 80), ∴ centripetal force in P : centripetal force in *p* in the ultimate R°. of P N : *p n*; or of Q R : *q r* (Art. 80), or of $\frac{\overline{QPA}^2}{PV} : \frac{\overline{qpa}^2}{pv}$, or by

reason of similar figures, (Art. 79) in the ultimate R°. of $\frac{\overline{QPA^2}}{PS} : \frac{\overline{qpa^2}}{ps}$.

Hence the centripetal forces in these similar points are also as the squares of the velocities directly, and the distances inversely; for the velocities are in the ultimate R°. of the arcs A P Q, *a p q* described in the same time.

Again, the centripetal forces at those similar points are also as the distances directly, and the squares of the periodic times inversely. For let A P Q, *a p q* no longer represent evanescent arcs described in the same time, but *similar* evanescent particles of the similar curves, described in the indefinitely small times T and *t*; also let V and *v* represent the velocities at P and *p*; A and *a* the whole areas of the similar figures; P and *p* the periodic times; then since A P Q, *a p q* may be considered as described uniformly, $V : v :: \frac{APQ}{T} : \frac{apq}{t} :: \frac{SP}{T} : \frac{sp}{t}$ (by Art. 79); but F at P : F at $p :: \frac{V^2}{SP} : \frac{v^2}{sp}$, $\therefore$ F at P : F at $p :: \frac{SP}{T^2} : \frac{sp}{t^2}$. But since T : P :: S Q A : A and $t : p :: sqa : a$, and that S Q A : sqa :: A : a (Art. 79), $\therefore T : t :: P : p$; hence F at P : F at $p :: \frac{SP}{T^2} : \frac{sp}{t^2} :: \frac{SP}{P^2} : \frac{sp}{p^2}$.

Hence since F varies as $\frac{V^2}{D}$ and as $\frac{D}{P^2}$ in similar figures, the preceding Cors. will apply to bodies describing similar parts of similar curves, having their centres of force similarly situated; for Ex. if the periodic time be as the n^{th} power of any homologous radius vectors, the forces will be reciprocally as the

$\overline{2n-1}^{th}$ power of any homologous radius vectors, and the contrary: and note, when distances are mentioned, the similar or homologous distances are always understood.

Prop. 4.—Cor. 9.

82. Let P A (*Fig.* 50) be an arc described in any time, P B the space fallen through in the same time by the force at P continued uniform; take P Q an evanescent arc, Q R the subtense parallel to P S, and complete the parallelogram; then the evanescent subtense Q R or P C is the space fallen through by the centripetal force, in the same time that the arc P Q is described (Art. 73). Let T and t represent the times of falling through P B and P C, or of describing the arcs P A, P Q; then since S varies as T^2, when F is given, $PC : PB :: t^2 : T^2 :: PQ^2 : PA^2 :: \frac{PQ^2}{PG} : \frac{PA^2}{PG}$; but $PC = \frac{PQ^2}{PG}$, $\therefore PB = \frac{PA^2}{PG}$ and $PB : PA :: PA : PG$.

DEDUCTIONS FROM PROP. 4 AND ITS CORS.

83. *Suppose a body to revolve uniformly in a circle; required the space through which it must fall, when acted upon by the centripetal force at the circumference continued uniform, in order to acquire the velocity it has in the circle.*

Let P B (*Fig.* 50) = required space, and suppose P A to be the arc uniformly described in the time of the body's falling through P B, then $PA = 2PB$; but (Cor. 9) $PB : PA :: PA : PG$, *i. e.* $PB : 2PB :: 2PB : PG$ or $2PS$, $\therefore PB = \frac{PS}{2} = \frac{1}{2}$ radius.

84. *Required the same in any curve.*

Let P O (*Fig.* 48) = required space, P V = chord of curvature, P Q an indefinitely small arc, and Q R (= P N) the subtense of the ∠ of contact; then since the velocities are as the spaces *uniformly* described in the same time, velocity in curve : velocity acquired through P N :: P Q : 2 P N, ∴

V^2 in curve : V^2 through P N :: PQ^2 : 4 PN^2; but
V^2 thro' P N : V^2 thro' P O, or V^2 in curve :: P N : P O

∴ $PQ^2 \times PN = 4\,PN^2 \times PO$, and $PO = \frac{PQ^2}{4PN} = \frac{PV}{4} = \frac{1}{4}$ of chord of curvature.

85. *Required the velocity and periodic time of a body revolving in a circle at the earth's surface.*

Let P Q (*Fig.* 50) be the arc described by the body in one second, P C the space fallen through by gravity in the same time, = $16\frac{1}{12}$ feet by experiment, = suppose to m, r = radius of the earth in feet; then (Cor. 9) m : P Q :: P Q : $2\,r$, ∴ P Q = $\sqrt{2\,m\,r}$; but P Q being the arc uniformly described in a given time, is a proper measure of the velocity, ∴ the required velocity = $\sqrt{2\,m\,r}$ feet per second.

Again to find the periodic time, we have $\sqrt{2\,m\,r}$ = arc described in one second; and if π = 3. 14159 &c. the whole circumference of the circle = $2\,\pi\,r$, ∴ since the motion is uniform, $\sqrt{2\,m\,r} : 2\,\pi\,r :: 1''$:

$$P.T = \frac{2\,\pi\,r}{\sqrt{2\,m\,r}} = \pi \times \sqrt{\frac{2\,r}{m}}$$ in seconds, r being expressed in feet.

Cor. 1. The velocity in miles = 4,92083 per second, and the P. T = $1^{hr}\ 24^{m}\ 27^{s}$.

Cor. 2. Hence if a body be projected from any point P on the earth's surface in a horizontal direction with the velocity of $\sqrt{2\,m\,r}$ feet in a second, it will revolve as a secondary round the earth; for suppose a body so to revolve, then at the point P it will have the same direction, the same velocity, and be acted upon by the same force as the projected body, $\therefore$ if the revolving body continue to move round the earth in a ⊙, the projected body must also revolve in the same manner.

Cor. 3. Hence also having given the radius of the circle described by any revolving body, and its velocity or periodic time, we can compare the centripetal force with that of gravity. For since by Prop. 4, F varies as $\frac{V^2}{R}$, $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$; call f the force of gravity, then will r = the earth's radius, and $v^2 = 2\,m\,r$, $\therefore$ F : gravity $:: \frac{V^2}{R} : 2\,m$.

Again since $F : f :: \frac{R}{P^2} : \frac{r}{p^2}$; call f the force of gravity, then will r = the earth's radius and $p^2 = \frac{2\,\pi^2\,r}{m}$, $\therefore$ F : gravity $:: \frac{R}{P^2} : \frac{m\,r}{2\,\pi^2\,r} :: \frac{R}{P^2} : \frac{m}{2\,\pi^2}$; where R must be expressed in feet, and P in seconds.

Cor. 4. We frequently meet in mathematical writers with the Equation $F = \frac{V^2}{R}$, *i. e.* F is said not to be proportional, but absolutely equal, to $\frac{V^2}{R}$; the Equation is deduced from the following supposition; we had in the last Cor. F : force of gravity $:: \frac{V^2}{R} : 2\,m$; now let the force of gravity be represented by its effect produced in a given time as 1″, or by $2\,m$; then $F : 2\,m :: \frac{V^2}{R} : 2\,m$, $\therefore F = \frac{V^2}{R}$. It must always $\therefore$ be kept in mind that when F is affirmed $= \frac{V^2}{R}$, it is done only on the supposition that gravity is represented not by unity, as is usually done, but by $2\,m$, its effect produced in 1″. If we represent gravity by unity, we shall then have $F : 1 :: \frac{V^2}{R} : 2\,m$; and in this case our Equation will be $F = \frac{V^2}{2\,m\,R}$, a conclusion deduced on the supposition that gravity is represented by 1. The difference then in the two equations $F = \frac{V^2}{R}$ and $F = \frac{V^2}{2\,m\,R}$ consists in this; that the former, $\frac{V^2}{R}$, is the measure of the centripetal force, estimated by the N°. of feet which it generates in 1″; *i. e.* it expresses a certain N°. of feet; whereas the latter, $\frac{V^2}{2\,m\,R}$, is an abstract N°., which is to the N°. 1 :: F : to the force of gravity; *i. e.* it is a certain multiple or part of the abstract N°. 1.* To shew the use of the two last

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* From what has been said, we may easily perceive the reason of the apparent contradiction in the above equations, viz. that since $F = \frac{V^2}{R}$, and also $F = \frac{V^2}{2\,m\,R}$, $\therefore F = 2\,m\,F$; to avoid however any seeming absurdity of this kind, it would perhaps be better to represent the force in the latter equation by F as before; and the force in the former equation, which is the *effect* of the force in the latter, by E; the equations would then have been $E = \frac{V^2}{R}$ and $F = \frac{V^2}{2\,m\,R}$, $\therefore E = 2\,m\,F$ a true equation, as may appear from hence: let gravity be represented by 1, and

Cors. let us apply them to the solution of the following Problems.

86\. 1. *Let a body revolve in the circle* M E D, *(Fig. 51) with a velocity acquired in falling through* M B *by gravity; required the Ratio of the centripetal force to that of gravity.*

Let V = velocity in curve, then $V^2 = 4m \times MB$ hence since F varies as $\frac{V^2}{R}$, we have as in first part of Corollary 3, F : gravity :: $\frac{4m \times MB}{MS}$: 2 m :: 2 M B : M S.

Cor. If the body be made to revolve uniformly in the ⊙ M E D by means of a weight fixed to a string; then we shall have the tension of the string arising from the centrifugal force of the body, to the tension arising from the same weight hanging freely, in the above R°. of 2 M B : M S.

2\. *Compare the force of gravity with the centrifugal force at the equator.*

Let P = time of the earth's revolving round its axis in seconds, R = radius of the earth in feet; then since F varies as $\frac{R}{P^2}$, we have as in 2d part of Cor. 3. Centrifugal force at Equator : Force of gravity :: $\frac{R}{P^2} : \frac{m}{2\pi^2}$

3\. *Given the moon's periodic time, and the radius of her orbit; to find how far she would fall in 1″, supposing her projectile motion to be destroyed.*

Let P = moon's periodic time, R = radius of her

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the centripetal force by F, and the effect of the force F by E; then gravity or 1 : F :: the effect of gravity in a given time or 2 m : the effect of F in the same given time, or E; ∴ E = 2 m F.

orbit, then since F varies as $\frac{R}{P^2}$ we have force of gravity at distance of moon : 1 :: $\frac{R}{P^2}$: $\frac{m}{2\pi^2}$, $\therefore$ F = $\frac{2\pi^2 R}{m P^2}$, $\therefore$ S = m F T^2 = $\frac{2\pi^2 R}{P^2}$.

4. *Required the periodic time of a body describing a conical surface.*

The body at B *(Fig. 52)* is retained in its orbit by three forces; gravity in direction S A, tension of the string in direction B S, and centrifugal force in direction A B; $\therefore$ the sides of the $\triangle$ S A B will represent them; hence centrifugal force or F : gravity or 1 :: A B : S A $\therefore$ F = $\frac{AB}{SA}$; hence since F varies as $\frac{R}{P^2}$ we have $\frac{AB}{SA}$: 1 :: $\frac{AB}{P^2}$: $\frac{m}{2\pi^2}$ $\therefore P^2 = \frac{2\pi^2 SA}{m}$ and $P = \pi\sqrt{\frac{2\,SA}{m}}$.

Cor. 1. Hence periodic time : T through 2 S A :: $\pi\sqrt{\frac{2\,SA}{m}}$: $\sqrt{\frac{2\,SA}{m}}$ :: π : 1 :: circumference of $\odot$: diameter.

Cor. 2. Required the periodic time when the tension of the string = 3 times the weight of the body. Let S B = L; then will S A, by Problem, = $\frac{L}{3}$, $\therefore$ F = $\frac{AB}{SA} = \frac{3\,AB}{L}$: hence $\frac{3\,AB}{L}$: 1 :: $\frac{AB}{P^2}$: $\frac{m}{2\pi^2}$, $\therefore P^2 = \frac{2\pi^2 SA}{m} = \frac{2\pi^2 L}{3m}$, and $P = \pi\sqrt{\frac{2L}{3m}}$.

PROPOSITION VI.

87. Let B P Q, *b p q*, (*Fig.* 53) be two indefinitely small arcs described in the times T, *t*; S and *s* the centres of force; S C P, *s c p*, the radius vectors, which ultimately bisect the chords B Q, *bq*, and ∴ also ultimately bisect the arcs B P Q, *b p q*, in P and *p*, (Art. 80); draw the tangents P R, *p r*, and the subtenses Q R, *q r* parallel to S P, *s p*; let also K P L be an arc described in the same time with *b p q*, and which shall be ultimately bisected by S P; then will its chord K L also be ultimately bisected by S P, and consequently P C, P N, *p c*, are ultimately the sagittæ of the arcs B P Q, K P L, *b p q*. Hence since K P L, *b p q* are arcs described in the same time,

$$PN : pc :: F \text{ at } P : F \text{ at } p; \text{ and by Cor. 2. Lem. II.}$$
$$PC : PN :: PQ^2 : PL^2 :: BPQ^2 : KPL^2 :: T^2 : t^2$$

$$\therefore PC : pc :: F \times T^2 : f \times t^2; \text{ and } F : f :: \frac{PC}{T^2} : \frac{pc}{t^2};$$ or the force in the middle of the arcs varies as

$$\frac{\text{sagittæ of those arcs}}{\text{time}^2 \text{ in which they are described.}}$$

This Prop. is general, being applicable to different bodies revolving in the same or different orbits, and round the same or different centres of force.

Prop. 6.—Cor. 1.

88. Let P Q and *p q* (*Fig.* 54) be two indefinitely small arcs, P R, *p r* tangents at P and *p*; Q R, *q r* subtenses parallel to S P, *s p*; then Q R, *q r* are ultimately = the sagittæ of two arcs whose middle points are P, *p* (Art. 80) or the sagittæ of double the arcs P Q, *p q*; also the time of describing 2 P Q is

ultimately proportional to the time of describing PQ; hence F at P : F at p :: $\frac{QR}{T^2..2PQ}$: $\frac{qr}{T^2..2pq}$ :: $\frac{QR}{T^2..PQ}$: $\frac{qr}{T^2..pq}$ :: (since in the same curve the areas are proportional to the times) $\frac{QR}{SPQ^2}$: $\frac{qr}{Spq^2}$ :: $\frac{QR}{\frac{1}{2}SP.QT^2}$: $\frac{qr}{\frac{1}{2}Sp.qt^2}$ :: $\frac{2R}{SP.^2QT^2}$: $\frac{qr}{Sp.^2qt^2}$ *i.e.* the centripetal force, in different points of the same curve, is in the ultimate Ratio of $\frac{SP^2.QT^2}{QR}$ inversely.

Notes to Prop. 6.—*Cor.* 1.

89. $\frac{SP.^2QT^2}{QR}$ is called a solid, because it is of three dimensions; for $\frac{QT^2}{QR}$ being a third proportional to two lines Q R and Q T, must also itself be a line, and S P^2 is the product of two lines; ∴ $\frac{SP.^2QT^2}{QR}$ is the product of three lines, and is therefore analogous to the solid content of a parallelopepid, whose three adjacent sides are the three lines. Again, not only is the Ratio $\frac{SP.^2QT^2}{QR}$: $\frac{Sp.^2qt^2}{qr}$ a finite Ratio upon the coincidence of P and Q, but the terms of the R°. also are always finite; for S P^2 is finite, also since the △s S P Y, Q N T are ultimately similar S P^2 : S Y^2 :: Q T^2 : Q N^2 :: $\frac{QT^2}{QR}$: $\frac{QN^2}{QR}$; but the

limit of $\frac{QN^2}{QR}$ is the chord of curvature P V a finite line, $\therefore$ also the limit of $\frac{QT^2}{QR}$, and consequently of $\frac{SP^2 . QT^2}{QR}$, is finite.

90. Let A and B represent any two areas described in different curves in the times T and *t*; *a* and *b* the areas described in the same two curves in the same given time as 1″, then will $T : t :: \frac{A}{a} : \frac{B}{b}$. For since in the same curve, the areas are proportional to the times of description,

$$A : a :: T : 1'' \text{; for the same reason}$$
$$b : B :: 1'' : t$$

$\therefore Ab : Ba :: T : t$ or $\frac{A}{a} : \frac{B}{b} :: T : t$; *i. e.* the times of describing any two areas in different curves, are as those areas directly, and the areas in a given time inversely. If T and $t =$ the whole periodic times, A and B will represent the whole areas of the orbits; *i. e.* the periodic times in different curves are as the whole areas directly, and the areas in a given time inversely.

91. The formula for the centripetal force, given in the above Corollary, is only applicable to the finding the variation of the force, in different points of the same orbit, and does not extend to different curves; for in the proof of that Corollary, the area S P Q was assumed proportional to the time in which it was described; which is not true for different orbits (unless indeed in these equal areas are described in equal times.) The expression for the force may however be made general by Art. 90, thus,

In all cases F varies as $\frac{QR}{T^2 .... PQ}$; but by Art. 90

$T^2 \ldots . PQ$ or $T^2 \ldots .$ area SPQ is proportional to the $\frac{\text{area } SPQ}{\text{area dat. temp.}}$ in all cases, or to $\frac{2 \text{ area } SPQ}{\text{area dat. temp.}}$. Let $\therefore a$ = area described in a given time, then the force F will vary as $\frac{QR}{\frac{SP^2 \cdot QT^2}{a^2}}$, as $\frac{QR \times a^2}{SP^2 \cdot QT^2}$, which is a general expression applicable to different orbits round the same, or different centres of force. Or if A = whole area of curve, since $a : A :: 1'' : P.T$ (if P. T denote the periodic time), $\therefore a = \frac{A}{P.T}$, $\therefore$ F varies as $\frac{QR \times A^2}{SP^2 \cdot QT^2 \times P.T^2}$, which is also general for different orbits.

Prop. 6.—Cor. 2.

92. Draw SY, Sy *(Fig. 54)* perpendicular to the tangents at P and p; then since $SP \times QT = SY \times QP$, being each ultimately double of the $\triangle$ SQP, and that $Sp \times qt = Sy \times qp$ for the same reason, $\therefore$ F. at P : F. at p in the ultimate R°. of $\frac{QR}{SY^2 \times QP^2} : \frac{qr}{Sy^2 \times qp^2}$.

Notes to Prop. 6. Cor. 2.

93. For the reasons given in Art. 89, $\frac{SY^2 \times QP^2}{QR}$ is a solid, and it is also finite upon the coincidence of Q and P; for SY^2 is finite, and $\frac{QP^2}{QR}$ is ultimately = chord of curvature PV, a finite line.

94. The above Corollary is only applicable to different points of the same curve, for the reasons given in Art. 91; but it may be made general by the me-

thod pursued in the former Corollary, from which it appears that the centripetal force will vary ultimately as $\frac{QR \times a^2}{SY^2 \times QP^2}$, or as $\frac{QR \times A^2}{SY^2 \times QP^2 \times \overline{P.T}^2}$.

Prop. 6.—Cor. 3.

95. By Cor. 2, F. at P : F. at p :: $\frac{QR}{SY^2 \times QP^2}$: $\frac{qr}{Sy^2 \times qp^2}$ ultimately, but $\frac{QP^2}{QR}$ is ultimately = chord of curvature at P = PV, and $\frac{qp^2}{qr} = pv$, ∴ F. at P : F. at p :: $\frac{1}{SY^2 \times PV}$: $\frac{1}{Sy^2 \times pv}$.

Notes to Prop. 6.—Cor. 3.

96. In general F varies as $\frac{a^2}{SY^2 \times PV}$, or as $\frac{A^2}{SY^2 \times PV \times \overline{P.T}^2}$.

97. From this Cor. may easily be deduced De Moivre's expression for the centripetal force. For let PN *(Fig. 55)* be the curve, PF the diameter of curvature, and PC = radius of curvature = R, the rest as before; then by similar △s, SP : SY :: PF (2 R) : PV = $\frac{2R \times SY}{SP}$, ∴ F varies inversely as $\frac{SY^3 \times R}{SP}$; which expression may be made general in the same manner as the rest.

Prop. 6.—Cor. 4.

98. By Cor. 3, F. at P : F. at p :: $\frac{1}{SY^2 \times PV}$

$: \frac{1}{Sy^2 \times pv}$, but (Cor. 1. Prop. 1) $SY^2 : Sy^2 ::$ $V.^2$ at p : $V.^2$ at P; $\therefore$ F. at P : F. at p :: $\frac{V.^2 \text{ at } P}{PV}$: $\frac{V.^2 \text{ at } p}{pv}$, or the centripetal force varies as $\frac{V^2}{\text{ch. curv}^e}$.

Notes to Prop. 6.—Cor. 4.

99. Now let there be two different orbits, and let the areas described in a given time in each be A and a; then, Art. 96, F. at P : F. at p :: $\frac{A^2}{SY^2 \times PV}$: $\frac{a^2}{Sy^2 \times pv}$; but in different orbits $SY^2 : Sy^2 ::$ $\frac{A^2}{V.^2 \text{ at } P} : \frac{a^2}{V.^2 \text{ at } p}$ (Art. 66); $\therefore$ F. at P : F. at p :: $\frac{V.^2 \text{ at } P}{PV} : \frac{V.^2 \text{ at } p}{pv}$, the same as before. Hence the formula $\frac{V^2}{PV}$ for the centripetal force in Cor. 4 is general, and applicable either to one or different orbits, round the same or different centres of force, and the reason why a general expression should be deduced from one that is not general, is obvious from the method of proof observed in this Note.

100. The Equation $F = \frac{V^2}{\frac{1}{2} PV}$ is of frequent occurence in mathematical writers; this Equation is deduced from the following supposition; $F : f :: \frac{V^2}{PV} : \frac{v^2}{pv}$; where f, v, and pv are some known standard quantities; let this standard force f be the force of

N.B.

gravity acting upon a body revolving at the earth's surface; then $v^2 = 2\,m\,r$, and $p\,v$ = earth's diameter $= 2\,r$, $\therefore$ F : force of gravity $:: \frac{V^2}{PV} : \frac{2\,m}{2}$. Now let the force of gravity be represented by $2\,m$; then $F : 2m :: \frac{V^2}{P\ V} : \frac{2\,m}{2} \therefore F = \frac{2\,V^2}{P\,V} = \frac{V^2}{\frac{1}{2}PV}$. It must always $\therefore$ be kept in mind that when F is said to $= \frac{V^2}{\frac{1}{2}PV}$, it is upon the supposition that gravity is represented by its effect produced in 1″ or by $2\,m$; and consequently that $\frac{V^2}{\frac{1}{2}PV}$ is the N°. of feet that a body would fall in 1″ by the uniform action of the force F, see Cor. 4. Art. 85.

Or the Equation may be thus deduced. In general $V^2 = 4\,m\,F\,S = 4\,m\,F \times \frac{P\,V}{4}$ (Art. 84) $= 2\,m\,F \times \frac{P\,V}{2}$; but gravity or 1 : F :: the effect of gravity or $2\,m$: the effect of $F = 2\,m\,F = E \therefore V^2 = E \times \frac{P\,V}{2}$ and $E = \frac{V^2}{\frac{1}{2}\,P\,V}$.

Cor. Hence $F = \frac{V^2}{m.\,P\,V}$, gravity being represented by 1; and $F = \frac{V^2}{\frac{1}{2}\,P\,V}$, gravity being represented by $2\,m$, or $F = \frac{V^2}{m.\,P\,V}$ and $E = \frac{V^2}{\frac{1}{2}\,P\,V}$.

INTRODUCTORY ARTICLES TO THE REMAINING PARTS OF THIS SECTION.

101. *If a body, urged by any centripetal force, is moved in any manner; and another body ascends or descends in a right line; and their velocities are equal in any one case of equal altitudes, their velocities will be equal at all equal altitudes.*

NEWT. LIB. I., PROP. 40.

Let any body descend from A (*Fig.* 56) through D, E, to the centre C; and let another body be moved from V in the curve line V I K *k*. With the centre C, at any intervals, let the concentric circles D I, E K be described, meeting the right line A C in D and E, and the curve line V I K in I and K. Let I C be joined meeting K E in N; and let the perpendicular N T be drawn to I K; and let the interval D E or I N of the circumferences of the circles be very small; and let the bodies have equal velocities in D and I. Since the distances C D, C I are equal, the centripetal forces in D and I will be equal. Let these forces be expressed by the small equal lines D E, I N; and if one force I N is resolved into two N T and I T; the force N T, by acting in the direction of the line N T, perpendicular to I T K the path of the body, will not change the velocity of the body in that path, but will only draw the body from its rectilinear course, and make it turn aside continually from the tangent of the orbit, and proceed in the curvilinear path I T K *k*. In producing this effect, that whole force will be employed: but the other force I T, by acting in the direction of the course of the body, will be wholly employed in accelerating it, and in a very small given time will produce an acceleration proportional to itself. Therefore the accelerations of the bodies in D and I, produced in equal times (if the limits of the ratios of the nascent lines D E, I N, I K, I T, N T are taken) are as the lines D E, I T; but in unequal times, are as those lines and the times jointly. But the times in which D E and I K are described, because of the equal velocities, are as the spaces described D E and I K; and therefore the accelerations, in the course of the bodies through the lines D E and I K, are as D E and I T, D E and I K jointly; that is, as DE^2 and the rectangle $IT \times IK$. But the rectangle $IT \times IK$ is equal to IN^2, that is equal to DE^2; and therefore equal accelerations are generated in the transit of the bodies from D and I to E and K: therefore the velocities of the bodies in E and K are equal: and by the same argument they will always be found equal in all subsequent equal distances. Which was to be demonstrated.

By the same argument, bodies with equal velocities, and equally distant from the centre, will be equally retarded in their ascent to equal distances. Which was to be demonstrated.

Hence the following Corollary.

Cor. Let C be the centre of force, A the point from which a body must fall by the action of the force to acquire the velocity in the curve at V, C D and C I equal distances from the centre C in the straight line and curve; v = velocity at I, $CI = x$, F = force in direction I C, then will $v\dot{v}$ vary as $-F\dot{x}$; for v, $\dot{v}$, F and $\dot{x}$ are the same, both in the curve and straight line. Hence, according to whatever law the velocity of the body descending in the right line V C may vary, in the same manner will the velocity in the curve also vary.

102. *To find the fluxional expression for the law of the force, supposing a body to revolve round a fixed centre.*

Let y = distance of the body from the centre of force, p = perpendicular upon the tangent, F = force, and v = velocity at the distance y; then v^2 varies as $\frac{1}{p^2}$ $\therefore$ $v\dot{v}$ varies as $-\frac{\dot{p}}{p^3}$; but Cor., Art. 101, $v\dot{v}$ varies as $-$ F $\dot{y}$, $\therefore$ F $\dot{y}$ varies as $\frac{\dot{p}}{p^3}$ and F varies as $\frac{\dot{p}}{p^3\dot{y}}$.

Or the same immediately follows from Prop. 6, Cor. 3, for F varies as $\frac{1}{SY^2 \times PV}$ as $\frac{1}{p^2 \times \frac{2p\dot{y}}{\dot{p}}}$, or as $\frac{\dot{p}}{p^3\dot{y}}$.

This expression is evidently only applicable to the comparison of the force at different points of the same orbit. In general F varies as $\frac{a^2 \times \dot{p}}{p^3\dot{y}}$.

Ex. 1. *Required the law of the force in the hyperbolic spiral.*—Here $p = \frac{ay}{\sqrt{a^2+y^2}}$ $\therefore$ $\frac{1}{p^2} = \frac{1}{y^2} + \frac{1}{a^2}$ $\therefore$ $\frac{\dot{p}}{p^3}$ varies as $\frac{\dot{y}}{y^3}$ and F varies as $\frac{\dot{p}}{p^3\dot{y}}$ or as $\frac{1}{y^3}$.

✓ Ex. 2. *Required the same in the spiral of Archimedes.*—Here $p = \frac{y^2}{\sqrt{b^2+y^2}}$, $\therefore$ $\frac{1}{p^2} = \frac{b^2}{y^4} + \frac{1}{y^2}$, $\therefore$ $\frac{2\dot{p}}{p^3} = \frac{4b^2\dot{y}}{y^5} + \frac{2\dot{y}}{y^3}$, $\therefore$ F $\doteq$ $\frac{\dot{p}}{p^3\dot{y}}$ or as $\frac{2b^2}{y^5} + \frac{1}{y^3}$.

Ex. 3. *Required the same in the involute of a circle.*—Let r = radius of the ⊙, then by the nature of the curve $p^2 = y^2 - r^2$, $\therefore$ $\frac{1}{p^2} = \frac{1}{y^2 - r^2}$, and $\frac{\dot{p}}{p^3\dot{y}}$

$= \frac{y}{\overline{y^2 - r^2}|^2}$ as $\frac{y}{p^4}$.

Ex. 4. *Required the same when the square of the velocity is proportional to the logarithm of the distance.* Here $v^2 = \log. y$, $\therefore \frac{1}{p^2} = \log. y$, $\therefore \frac{\dot{p}}{p^3 \dot{y}} = -\frac{1}{y}$; the force $\therefore$ is repulsive, and varies inversely as the distance.

103. *The squares of the velocity of bodies revolving in any curve, are in the joint Ratio of the accelerating forces, and chords of curvature.*

For $F : f :: \frac{V^2}{PV} : \frac{v^2}{pv}$, $\therefore V^2 : v^2 :: F \times PV : f \times pv$.

104. *To compare the velocity in any point of the curve, with the velocity of a body revolving in a circle at the same distance.*

Let F and f be the forces in the curve and $\odot$; V and v their velocities; then $F : f :: \frac{V^2}{PV} : \frac{v^2}{pv}$; but since the distances are equal, $F = f \therefore \frac{V^2}{PV} = \frac{v^2}{pv}$ and $V^2 : v^2 :: PV : pv$.

Cor. Let y = distance from the centre of force, p = perpendicular on the tangent, then if for PV, pv, we substitute their values, we shall have $V^2 : v^2 :: \frac{2p\dot{y}}{\dot{p}} : 2y :: \frac{\dot{y}}{y} : \frac{\dot{p}}{p}$.

105. *If a body revolve in a curve of any kind round a centre of force, to compare the $\angle^r$ velocity of the perpendicular upon the tangent, with that of the radius vector.*

Let P, Q (*Fig.* 8) be two points in the curve indefinitely near to each other, to which the tangents PY, Qy are drawn; let fall the perpendiculars SY,

Sy upon the tangents PY, Qy, and from P and Q draw PC, QC perpendicular to the curve at P and Q, which will meet in C the centre of curvature: then since PC, QC are respectively parallel to YS, yS, the $\angle PCQ = \angle YSy$; hence $\angle^r$ velocity of perpendicular : $\angle^r$ velocity of distance :: $\angle YSy$: $\angle PSQ :: \angle PCQ : \angle PSQ :: \frac{QP}{PC} : \frac{QT}{SP} :: \frac{CP}{CP} : \frac{PO}{SP} :: SP : PO :: 2SP : PV :: 2y : \frac{2p\dot{y}}{\dot{p}} :: \frac{\dot{p}}{p} : \frac{\dot{y}}{y}$.

Cor. Hence, and by Art. 104, V^2 in curve : V^2 in ⊙ at the same distance :: $\angle^r$ velocity of distance : $\angle^r$ velocity of perpendicular; and ∴ the velocity in the curve = velocity in the ⊙ at the same distance, when the $\angle^r$ velocity of the distance = the $\angle^r$ velocity of the perpendicular.

106. *The angular velocity in any curve is as the area described in a given time directly, and the square of the distance inversely.*

Let PSQ, psq, (*Fig.* 57) be two indefinitely small $\angle$s; A and a the areas described about S and s in the same given time, then $\angle^r$ velocity about S : $\angle^r$ angular velocity about s :: $\angle PSQ : \angle psq :: \frac{QT}{SP} : \frac{qt}{sp} :: \frac{SP.QT}{SP^2} : \frac{sp \times qt}{sp^2} :: \frac{A}{SP^2} : \frac{a}{sp^2}$

Cor. In the same curve $A = a$, ∴ $\angle^r$ velocity $\propto \frac{1}{\text{dist.}^2}$

107. *To find the variation of the paracentric velocity in any curve.*

Let PQ (*Fig.* 58) represent the velocity in the curve; draw QT perpendicular to SP, then will PT represent the velocity towards the centre; to find which, put $SP = y$, $SY = p$, then $SP : PY ::$ $PQ : PT = \frac{PQ \times PY}{SP} \stackrel{.}{-} \frac{\frac{1}{p} \times \sqrt{y^2 - p^2}}{y}$, as $\frac{\sqrt{y^2 - p^2}}{py}$.

108. *Required the rate at which the linear velocity decreases in any curve.*

Let $SP = y$, $SY = p$, $v =$ velocity in curve at P, then since $v \stackrel{.}{-} \frac{1}{p}$, $- \dot{v} \stackrel{.}{-} - \frac{\dot{p}}{p^2}$ or $\dot{v} \stackrel{.}{-} \frac{\dot{p}}{p^2}$: from the equation to the curve get a value of p in terms of y, and consequently a value of $\frac{\dot{p}}{p^2}$ in terms of y and $\dot{y}$; but $\sqrt{y^2 - p^2} : p :: PT$ or $\dot{y} : QT = \frac{p\dot{y}}{\sqrt{y^2 - p^2}}$ $\therefore \frac{py\dot{y}}{\sqrt{y^2 - p^2}} = SP \times QT =$ area described in a given time $= 1$, $\therefore \dot{y} = \frac{\sqrt{y^2 - p^2}}{py}$; substitute this value of $\dot{y}$ in the proportional equation $\dot{v} \stackrel{.}{-} \frac{\dot{p}}{p^2}$, and the thing required is done.

109. *Required the rate at which the $\angle^r$ velocity decreases in any curve.*

Let α represent the $\angle^r$ velocity, then $\alpha \stackrel{.}{-} \frac{1}{y^2}$ $\therefore$ $- \dot{\alpha} \stackrel{.}{-} - \frac{\dot{y}}{y^3}$ or $\dot{\alpha} \stackrel{.}{-} \frac{\dot{y}}{y^3}$; but by the last Art. $\dot{y} = \frac{\sqrt{y^2 - p^2}}{py}$ $\therefore \dot{\alpha} \stackrel{.}{-} \frac{\sqrt{y^2 - p^2}}{py^4}$.

110. Supposing a body to revolve about a centre of force, and the motion in the curve to be resolved into two, one in the direction of the radius vector, and the other perpendicular to it, it is evident that that part of its motion, which is perpendicular to the radius vector, will give the body a tendency to recede from the centre. This tendency of the body to recede from the centre, in consequence of its rotation round it, is called the *centrifugal* force, and the space by which it thus recedes, in an indefinitely small given time, is the measure of this force.

Thus let P Q (*Fig.* 57) be an arc described in an indefinitely small given time, S the centre of force; resolve P Q into P T and T Q, and with S as centre and S Q as radius describe the circular arc Q x. Now since P Q represents the whole motion of the body, P T will represent that part of it which is towards the centre; and by this *alone* the body would be found at the distance S T from the centre at the end of the given time; but in consequence of the motion T Q perpendicular to S P, it is really found at Q at the end of the given time, and at a distance from the centre $=$ S Q or S x. In consequence ∴ of the perpendicular motion T Q, the body has receded from the centre through a space $=$ T x, which ∴ by the definition is a measure of the centrifugal force.

111. Strictly speaking the term *force* applied to this tendency of a body to recede from the centre in consequence of its rotation round it is inaccurate; it being merely the effect of that property in all matter of preserving in its rectilineal direction; it is ∴ denominated a force merely because we must employ a centripetal force to balance it, just as we suppose a *resisting* vis inertiæ because we must employ force to move a body.

112. From the above definition of a centrifugal force it follows (1) That if a body revolve in a circle, the centripetal and centrifugal forces are equal; for T P (*Fig.* 59) is the space through which the body

recedes from the centre in consequence of the perpendicular motion T Q, and ∴ represents the centrifugal force; also P T taken in a contrary direction represents the effect of the centripetal force ∴ &c.

Or the same conclusion may be deduced from considering that the body always continues at the same distance from the centre, and ∴ through whatever space it must recede from the centre in consequence of the centrifugal force; through the same space must it approach the centre in consequence of the centripetal. (2) That if a body revolving in any curve come to an apse, it will, after that, approach to, or recede from the centre, according as the centripetal is greater, or less, than the centrifugal force. For let P Q (*Fig.* 60) be the curve, P the apse, P A a ⊙ described with S as centre and S P as radius, and which falls without the curve P Q; then by constructing the figure as before, we shall have T x to represent the centrifugal force, and P T the centripetal, but since S A is greater than S Q, P T is greater than T x, *i. e.* when the body approaches the centre from an apse, centripetal force is greater than centrifugal, ∴ conversely, &c.

But if ⊙ P A (*Fig.* 61) falls within the curve, *i. e.* if the body recedes from the centre, T x is greater than P T *i. e.* centrifugal force is greater than centripetal, ∴ &c.

Or the same conclusion may be deduced from considering that since the whole motion towards the centre is the effect of the centripetal force, and the whole motion from it the effect of the centrifugal, the body must approach to, or recede from the centre according as the first is greater or less than the second. (3) That if the body be not at an apse, *i. e.* if the direction of the body's motion be oblique to the radius vector, the body's approach to, or recess from, the centre, does not depend upon the centripetal force being greater or less than the centrifugal; for in this case P T (*Fig.* 62) $= \text{P}\,y + y\,\text{T} = \text{P}\,y + \text{Q R}$, *i. e.* the motion directly towards the centre is made up of the motion Q R in that direction arising from the action of the centripetal force together with that part of the tangential motion represented by P y which is in the direction P S; hence in consequence of this tangential motion the body may approach to the centre S, even though the centrifugal force be greater than the centripetal, as is represented in the figure, and the contrary. (4) That in all cases the centrifugal is equal and opposite to the centripetal force of a body revolving in a circle at the same distance and with the same $\angle^r$ velocity; for if x Q represent a circular arc described in the same given time in which the arc P Q is described, x T will be a measure of the centripetal force in that circle, but T x has been shewn also to represent the centrifugal force of the body revolving in the curve P Q.

113. *The centrifugal force in different points of different curves is proportional to the square of the area described in a given time directly, and the cube of the distance inversely.*

For centrifugal force at P (*Fig.* 57) : D°. at p ::

$$Tx : tx :: \frac{QT^2}{SP} : \frac{qt^2}{sp} :: \frac{SP^2 \times QT^2}{SP^3} : \frac{sp^2 \times qt^2}{sp^3}$$

$$:: \frac{A^2}{\text{Dist.}^3} : \frac{a^2}{\text{Dist.}^3}$$

Cor. In the same curve A $= a$, *i. e.* in different points of the same curve, the centrifugal force varies as $\frac{1}{\text{Dist.}^3}$.

114. *To compare the centripetal and centrifugal forces in any curve.*

Centripetal : centrifugal force :: Q R : T x ::

$$\frac{PQ^2}{PV} : \frac{QT^2}{2SP} :: \frac{SP^2}{PV} : \frac{SY^2}{2SP} \text{ (by similar } \triangle^s\text{)}, ::$$

$2\,SP^3 : SY^2 \times PV$.

Cor. From this proportion may be deduced many of the conclusions in Art. 112. For instance, in circular orbits centripetal force = centrifugal; for in that case centripetal : centrifugal force :: $2\,SP^3$: $SP^2 \times 2\,SP$ :: $2\,SP^3$: $2\,SP^3$. Also if at an apse centripetal force be greater than centrifugal, the body will approach the centre; for centripetal force : centrifugal :: $2\,SP^3$: $SY^2 \times PV$; *i. e.* at an apse :: $2\,SP$: PV, ∴ if centripetal force be greater than centrifugal, $2\,SP$ is greater than PV, or SP greater than $\frac{1}{2}PV$; hence if PB (*Fig.* 63) be the curve, $PO = \frac{1}{2}PV$, and if a ⊙ PC be described with centre S and radius SP, it will fall without the ⊙ of curvature PA, and ∴ also without the curve PB, *i. e.* the body will approach to the centre, and the contrary.

PROPOSITION VII.

Notes to Prop. 7.

115. If A = the whole area of the circle, and a = area described in a given time, we have, as in Art. 91, the centripetal force to vary, in general, as $\frac{a^2 \times AV^2}{SP^2 \times PV^3}$ or as $\frac{A^2 \times AV^2}{SP^2\,PV^3 \times P.T^2}$, which is true for different ⊙s having the same or different centres of force.

116. If the centre of force S (*Fig.* 64) be without the circle, $\frac{1}{SP^2 \times PV^3}$, which expresses the law of the force, is positive, while the body moves from B through P to A; but at A and B, PV vanishing, the force becomes infinite. From A through V and P′ to B, PV lying the contrary way to what it did in

the superior part of the orbit, the expression for the force becomes negative; the centre ∴ repels the body.

117. To prove the Prop. fluxionally, let S P (*Fig.* 55) $= y$, P V $= c$, S Y $= p$, P F $= b$; then P S. S V = A S. S B = some constant quantity, $= a^2$, *i. e.* $y \times \overline{c - y} = a^2$, $\therefore c = \dfrac{a^2 + y^2}{y}$. Also by similar $\triangle^s$ $y : p :: b : \dfrac{a^2 + y^2}{y}$ $\therefore p = \dfrac{a^2 + y^2}{b}$ and $\dfrac{1}{p^2} = \dfrac{b^2}{\overline{a^2 + y^2}^2}$. $\therefore \dfrac{\dot{p}}{p^3} = \dfrac{2 b^2 y \dot{y}}{\overline{a^2 + y^2}^3}$ and $\dfrac{\dot{p}}{p^3 \dot{y}} = \dfrac{b^2 y}{\overline{a^2 + y^2}^3}$, as $\dfrac{b^2 y^3}{y^2 \times \overline{a^2 + y^2}^3}$, as $\dfrac{b^2}{y^2 \times c^3}$.

118. By Arts. 103 and 104, the velocity in the curve $= \dfrac{1}{\text{S P. P V}}$; and the velocity in the curve at P : velocity in ⊙ at same distance :: $\sqrt{\tfrac{1}{2}\text{P V}} : \sqrt{\text{S P}}$. If S be in the circumference of the ⊙, the R°. becomes that of $1 : \sqrt{2}$.

119. By Art. 114. Centripetal force : centrifugal :: 2 S P. A V^2 : P V^3.

Notes to Prop. 7.—Cor. 2.

120. If the periodic times be not equal, then neither are the areas described in a given time round the two centres equal; ∴ in that case, by Art. 115, F round S : F round R :: $\dfrac{\text{R P}^2.\ \text{S P}}{\text{P. T. round S}^2} : \dfrac{\text{S G}^3}{\text{P. T. round R}^2}$; since A V^2 and the whole areas are the same in both cases.

121. Suppose R (*Fig.* 65) to be in the centre of the circle, and S to be at V in the circumference; to compare the forces round each centre, the periodic times being the same. Since the whole areas and

periodic times are the same in both cases, F $\doteq \frac{1}{SP.^2\, PV^3}$, ∴ F. round R : F. round V :: $\frac{1}{RP.^2\, PT^3}$: $\frac{1}{PV^5}$ :: $\frac{1}{\frac{1}{4} PT^5}$: $\frac{1}{PV^5}$ :: $PV^5 : \frac{1}{4} AV^5$.

PROPOSITION VIII.

Notes to Prop. 8.

122. Since F $\doteq \frac{CP^2}{2\, PM^3 \times SP^2}$, and that SP^2 is infinite, it might be inferred that force was infinitely small; the contrary however will appear from the general solution. For, in general, F varies as $\frac{QR.\, a^2}{SP^2\, QT^2}$; but $V^2 = \frac{4\, a^2}{SY^2}$ ∴ $a^2 = \frac{V^2.\, SY^2}{4}$. Now let b = velocity in direction A C, which is constant (Art. 125), ∴ $V^2 : b^2 :: PR^2 : QT^2 :: SP^2 : SY^2$ ∴ $V^2 = \frac{b^2\, SP^2}{SY^2}$ and $a^2 = \frac{b^2\, SP^2}{SY^2} \times \frac{SY^2}{4} = \frac{b^2\, SP^2}{4}$ ∴ $\frac{QR.\, a^2}{SP^2.QT^2} = \frac{CP^2}{2PM^3.SP^2} \times \frac{b^2\, SP^2}{4} = \frac{b^2\, CP^2}{8\, PM^3}$ a finite quantity when P M is finite.

123. By Art. 103, the velocity in the curve at P $\doteq \frac{1}{PM}$.

124. By Art. 114, centripetal force at P : centri-

fugal :: 2 S P^3 : SY^2 × P V :: 2 S P × C P^2 : P M^2 × P M, *i. e.* centrifugal force is nothing, as also appears from the definition of a centrifugal force in Art. 110.

125. To find the fluxional expression for the law of the force, supposing this force to act in parallel lines.

Let A B (*Fig.* 66) $= x$, B P $= y$, P T $= \dot{x}$, T R $= \dot{y}$, $b =$ velocity in the direction A B, which in the same curve will be constant, since the force in the direction of the ordinate does not affect the motion of the body in the direction of the abscissa: $v =$ velocity in the direction of the ordinate B P, and F = force in the direction P B; then $\dot{x} : \dot{y} :: b : v = \frac{b\dot{y}}{\dot{x}} \therefore \dot{v} = \frac{b\ddot{y}}{\dot{x}}$ and $v\dot{v} = \frac{b^2\dot{y}\ddot{y}}{\dot{x}^2}$; but $-v\dot{v} \doteq F\dot{y}$ $\therefore F \times \dot{y} \doteq -\frac{b^2\dot{y}\ddot{y}}{\dot{x}^2}$ and F is as $-\frac{b^2\ddot{y}}{\dot{x}^2}$ or as $-\frac{\ddot{y}}{\dot{x}^2}$; or if P Q be an arc described in an indefinitely small *given* time $\dot{x}$ is constant, and $F \doteq -\ddot{y}$.

Or thus. By Prop. 6, $F \doteq \frac{QR}{T^2 \ldots PQ}$ but $T^2 \ldots PQ$ or $PT \doteq \frac{PT^2}{b^2} \therefore F \doteq \frac{QR \times b^2}{PT^2}$, or as $\frac{b^2 \times -\ddot{y}}{\dot{x}^2}$ as before.

126. To prove the Prop. fluxionally, put P M = y, C M $= x \therefore y = \sqrt{r^2 - x^2}$ and $\dot{y} = \frac{-x\dot{x}}{\sqrt{r^2 - x^2}} = \frac{-x\dot{x}}{y} \therefore \ddot{y} = \frac{x\dot{x}\dot{y} - y\dot{x}^2}{y^2} = \frac{x\dot{x} \times -\frac{x\dot{x}}{y} - y\dot{x}^2}{y^2}$ $= \frac{-\dot{x}^2 \times \overline{x^2 + y^2}}{y^3} \therefore F \doteq -\frac{\ddot{y}}{\dot{x}^2}$, as $\frac{1}{y^3}$.

SCHOLIUM.

Introductory Article to Scholium.

127. *Lemma.*—Let P O (*Fig.* 67) be the diameter of curvature of the conic section D P L, C the centre, C D the ½ conjugate diameter produced to meet P O in F, then will $PO \propto PA^3$.

For $CD \propto \frac{1}{PF}$ ∴ CD^2 is as $\frac{1}{PF^2}$ and $\frac{2\,CD^2}{PF}$ or P O as $\frac{1}{PF^3}$; but by conics $PA \propto \frac{1}{PF}$ ∴ PA^3 is as $\frac{1}{PF^3}$ and P O as PA^3.

Cor. If the distance betwixt the foci of the ellipse increase, PO still $\propto PA^3$; if ∴ this distance become infinite or the ellipse migrate into a parabola, $PO \propto PA^3$, and hence the Prop. is general.

Scholium.

128. Let L P D be any conic section, P V the chord of curvature perpendicular to the axis, then

$$QT^2 : PR^2 :: PM^2 : PA^2$$

$$\therefore \frac{QT^2}{QR} : \frac{PR^2}{QR} \text{ or } PV :: PM^2 : PA^2$$

$$\text{but } PV : PO :: PM : PA$$

$$\therefore \frac{QT^2}{QR} : PO :: PM^3 : PA^3 \therefore \frac{QT^2}{QR} = \frac{PO}{PA^3}$$

$\times PM^3 \propto$ (by Art. 127) $PM^3 \therefore \frac{SP.^2\,QT^2}{QR} \propto$

$P M^3$; and $F \propto \frac{1}{PM^3}$.

The same fluxionally.

In parabola $y^2 = 2ax \therefore y\dot{y} = a\dot{x}$ and $\dot{y} = \frac{a\dot{x}}{y} \therefore -\ddot{y} = \frac{a\dot{y}\dot{x}}{y^2} = \frac{a^2\dot{x}^2}{y^3} \therefore F \propto -\frac{\ddot{y}}{\dot{x}^2}$ as $\frac{a^2}{y^3}$ as $\frac{1}{y^3}$.

In ellipse and hyperbola $y = \frac{b}{a} \times \sqrt{a^2 - x^2}$, $\therefore$

$$\dot{y} = \frac{b}{a} \times \frac{-x\dot{x}}{\sqrt{a^2 - x^2}} = \frac{b}{a} \times \frac{-x\dot{x}}{\frac{a}{b} \times y} = \frac{b^2}{a^2} \times \frac{-x\dot{x}}{y},$$

$$\therefore \ddot{y} = \frac{b^2}{a^2} \times \frac{x\dot{x}\dot{y} - \dot{x}^2 y}{y^2} = \frac{b^2}{a^2} \times \frac{-\frac{b^2}{a^2}.x^2\dot{x}^2 - \dot{x}^2 y^2}{y^3}$$

$$\propto \frac{-\dot{x}^2 \times \frac{b^2}{a^2}.x^2 + y^2}{y^3};$$ but since $y^2 = \frac{b^2}{a^2}.\overline{a^2 - x^2}$,

$\therefore \frac{b^2}{a^2}.x^2 + y^2 = b^2, \therefore \ddot{y} \propto \frac{-\dot{x}^2 \times b^2}{y^3}$ and $F \propto -\frac{\ddot{y}}{\dot{x}^2}$, as $\frac{1}{y^3}$.

PROPOSITION IX.

Introductory Article to Proposition 9.

129. The curve which cuts all its radii, drawn from a fixed point, in a given ∠, is called the 'Equiangular Spiral.'

From this definition it follows (1) That if the radius vector revolve with a *uniform* angular velocity round the centre, in other words if the arcs $\alpha\,\beta$, $\alpha\,\gamma$, $\alpha\,\delta$, $\alpha\,\varepsilon$ &c. (*Fig.* 68) described with a given radius S α, and ∴ measuring the ∠s at the centre, increase in arithmetical progression, the Radii themselves will increase in geometrical progression. For let the equal ∠s A S B, B S C, C S D, &c. be taken *in the first place* indefinitely small, then since the ∠s at B, C, D, &c. are equal to each other, and also the ∠s at the centre, the figures S A B, S B C, S C D, &c. are ultimately similar △s, ∴ S A : S B :: S B : S C :: S C : S D :: &c.; but if quantities be in geometrical progression any equidistant terms in the series are also in geometrical progression; take ∴ $\alpha'\,\beta'$, $\beta'\,\gamma'$, $\gamma'\,\delta'$ &c. (*Fig.* 69) equal to each other, but of *finite* magnitude, then will S A′, S B′, S C′, S D′, &c. be equidistant terms of the former series S A, S B, S C, S D, &c. *they* are ∴ also in geometrical progression, or S A′ : S B′ :: S B′ : S C′ :: S C′ : S D′ :: &c. (2) That the ∠s at the centre are the measures of R^os^. of the corresponding radius vectors; for by the last S A′ : S B′ :: S B′ : S C′ ∴ S C′ : S A′ :: SB'^2 : SA'^2 ∴ the R°. S C′ : S A′ is double of the R°. S B′ : S A′, *i. e.* if arc $\alpha'\,\gamma'$ be taken double of $\alpha'\,\beta'$, the R°. S C′ : S A′ is double of the R°. S B′ : S A′; in like manner it may be shewn that if $\alpha'\,\delta'$ be taken triple of $\alpha'\,\beta'$, the R°. S D′ : S A′ is triple of the R°. S B′ : S A′ &c. hence the arcs $\alpha'\,\beta'$, $\alpha'\,\gamma'$, $\alpha'\,\delta'$ &c. or the ∠s at the centre represent the comparative magnitudes of the R^os^. of the coresponding radii; and ∴ if $\alpha'\,\beta'$ be assumed to represent the magnitude of the R°. S B′ : S A′ or of the R°. $\frac{SB'}{SA'}$: 1; $\alpha'\,\gamma'$ will represent the magnitude of the R°. S C′ : S A′ or $\frac{SC'}{SA'}$: 1; $\alpha'\,\delta'$ the magnitude of the R°. $\frac{SD'}{SA'}$: 1 &c. in other words if $\alpha\beta'$ be assumed to be the log. of $\frac{SB'}{SA'}$, $\alpha\gamma'$ will be the log. of $\frac{SC'}{SA'}$, $\alpha\delta'$ the log. of $\frac{SD'}{SA'}$, &c. and from this property it is that the spiral is frequently called the logarithmic spiral. (3) That the velocity with which the radius increases at any point is in proportion to the magnitude of the radius at that point; in other words that the fluxion of the radius is as the radius itself; for taking the ∠s at the centre indefinitely small as in the 1st case we have SA : SB :: SB : SC ∴ SA : SB :: SB − SA (BF) : SC − SB (CG); but the ultimate R°. of BF : CG is that of the fluxion of SA to the fluxion of SB; ∴ SA : SB :: fluxion of SA : fluxion of SB and so for the rest. (4) That the chord of curvature to any point of the spiral is double the radius vector at that point; for let S (*Fig.* 70) be the centre of the spiral, PQ an indefinitely small arc; from Q and P draw QO, PO perpendicular to the curve at Q and P respectively, which will meet in O the centre of curvature; take PV the chord of curvature passing through S, and join VQ, then since the ∠ OQA = ∠ OPQ, take from these the equal ∠s SQA, SPA, and the remainder the ∠ OQS = the remainder the ∠ OPS, and the ∠s at C are vertical ∠s, ∴ ∠ COP = ∠ QSC, but ∠ POC being at the centre is double the ∠ PVQ at the circumference, ∴ also ∠ PSQ = 2 ∠ PVQ; but ∠ PSQ = ∠ PVQ + ∠ SQV, ∴ ∠ SVQ = ∠ SQV and SQ or SP = SV, ∴ PV = 2 SP.

Prop. 9.

130. *Case* 1. Let PQ, *pq* (*Fig.* 71) be two indefinitely small arcs, and let us suppose in the first place the ∠ PSQ to be a given ∠, *i. e.* that the ∠ PSQ = ∠ *p*S*q*, then since the ∠s at S, P and R are respectively = the ∠s at S, *p* and *r*, the remaining ∠

S Q R = remaining ∠ Sqr; ∴ the figures S Q R P and $Sqrp$; Q R P T and $qrpt$; Q P T and qpt; S P Q and Spq are respectively similar to each other, and ∴ have their homologous sides proportional, ∴ QT : qt :: Q R : qr, ∴ $\frac{QT}{QR} = \frac{qt}{qr}$ and $\frac{QT^2}{QR}$: $\frac{qt^2}{qr}$ :: Q T : qt :: Q P : qp :: S P : sp, ∴ $\frac{SP^2 . QT^2}{RR}$: $\frac{Sp^2 . qt^2}{qr}$ :: $SP^3 : Sp^3$ :: F. at p : F. at P.

Case 2. Suppose the ∠ P S Q not to be = ∠ pSq; make in that case the ∠ $PS\pi$ = ∠ pSq, then by the first case $\frac{\pi\tau^2}{\pi\varrho} : \frac{qt^2}{qr}$ :: S P : Sp; but Q R : $\pi\varrho$ :: $QP^2 : \pi P^2$, *i. e.* by similar △s. :: $QT^2 : \pi\tau^2$, ∴ $\frac{\pi\tau^2}{\pi\varrho} = \frac{QT^2}{QR}$ ∴ $\frac{QT^2}{QR} : \frac{qt^2}{qr}$ :: S P : Sp as in the first case; and this is the meaning of Newton's expression, "if the ∠ P S Q is in any way changed."

131. To prove the Prop. fluxionally put S P = y, S Y = p; then $p : y$ in a given R°. :: $m : n$, ∴ $np = my$, and $\frac{1}{p^2} = \frac{n^2}{m^2 y^2}$ ∴ $\frac{\dot{p}}{p^3 \dot{y}} = \frac{n^2}{m^2 y^3} \propto \frac{1}{y^3}$.

132. By Arts. 103 and 104 the velocity in the curve $\propto \frac{1}{SP}$ and velocity in curve = velocity in a ⊙ at the same distance.

133. By Art. 114, centripetal force : centrifugal :: $SP^2 : SY^2$ :: $\overline{\text{rad.}}^2 : \overline{sin. \angle SPY}^2$, and ∴ in a constant R°. in the same spiral.

PROPOSITION X.

Notes to Prop. 10.

134. To make the Prop. general, let A = whole area of ellipse, a = area described in a given time, and P = periodic time, then by Art. 91 we have in general the centripetal force $\propto \frac{QR \times a^2}{SP^2 . QT^2}$ or as $\frac{QR \times A^2}{SP^2 . QT^2 \times P^2}$; *i. e.* in this case $\propto \frac{a^2}{AC^2 . CB^2} \times PC$, or $\propto \frac{AC^2 . CB^2}{AC^2 . CB^2} \times \frac{PC}{P^2} \propto \frac{PC}{P^2}$. Both which expressions are general, and true for bodies moving round different centres.

135. If different bodies revolve round the *same* centre, then at equal distances the forces will be equal; hence $\frac{a^2}{AC^2 . CB^2}$, or $\frac{a^2}{A^2}$ must be constant, ∴ when different bodies revolve round the same centre, the force $\propto$ C P.

136. Let φ represent the absolute force, then accelerating force $\propto \varphi \times PC \propto \frac{a^2}{A^2} \times PC$ ∴ $\varphi \propto \frac{a^2}{A^2}$.

137. To prove the Prop. fluxionally put $a = \frac{1}{2}$ axis major, $b = \frac{1}{2}$ axis minor, y = C P, p = P F = perpendicular on the tangent, then $p^2 = \frac{a^2 b^2}{a^2 + b^2 - y^2}$ and $\frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} - \frac{y^2}{a^2 b^2}$ ∴ $\frac{\dot{p}}{p^3 \dot{y}} = \frac{y}{a^2 b^2}$ and F $\propto y$.

138. $F \propto PC$, or as $\frac{V^2}{PV}$ $\therefore V^2 \propto PC \times PV$, as $PC \times \frac{CD^2}{PC}$, as CD^2 $\therefore V \propto CD$; this is true for bodies moving in the same or different orbits round the *same* common centre. But if bodies revolve round *different* centres of force $F \propto \frac{a^2}{A^2} \times PC$, or as $\varphi \times PC$; also $F \propto \frac{V^2}{PV}$ $\therefore V \propto \frac{a}{A} \times CD$, or as $\varphi^{\frac{1}{2}} \times CD$.

139. Velocity in ellipse : velocity in ⊙ at the same distance :: $\sqrt{\frac{2\,CD^2}{CP}} : \sqrt{2\,CP}$ $\therefore CD : CP$.

Cor. Hence the velocities in the ellipse and circle at the same distance are equal in four points of the ellipse. For through the extremities of the major and minor axes *(Fig. 72)* of the ellipse draw tangents which will form a rectangle. Join G C, H C, which produced will pass through I and K, since A F and H K, A B and H K are similar parallelograms, and ∴ about the same diameter; draw B M, which is bisected in L, consequently B M is an ordinate to the diameter P R; also B M is parallel to H K, ∴ Q D is a conjugate diameter to P R; and since $\angle BCP = \angle BCD$, $CP = CD = CR = CQ$, ∴ the velocities in the ellipse and ⊙ at the points P, D, R, Q are equal.

140. Centripetal force : centrifugal :: $PC^4 : AC^2 \cdot CB^2$. Hence these forces are equal when $PC^2 = AC, CB$ or when the distance from the centre is a mean proportional between the two ½ axes of the ellipse. To find this point geometrically; from C M (*Fig.* 73) cut off $CD = CB$; on D A as diameter describe a ½ ⊙ D E A, produce C B to meet it in E, and with C as center and C E as radius describe the

$\odot$ P P″ E; then will the centripetal force be = the centrifugal at the points P, P′, P″, P‴; for join C P then CP^2 (= CE^2 = A C. CD) = A C. C B, and the same may be proved of the other points.

Prop. 10.—*Cor.* 2.

141. That the periodic times in two similar ellipses round the same centre are equal appears from Cor. 8, Prop. 4. That they are also equal in two ellipses having a common axis major appears from hence: let E and *e* be two ellipses having a common axis major; then P. T. in E : P. T. in $e :: \frac{A}{a} : \frac{A'}{a'} :: \frac{AC.CB}{a} : \frac{AC.Cb}{a'} :: \frac{CB}{a} : \frac{Cb}{a'}$, but the areas described in a given time $\doteq$ Q P $\times$ S Y, as velocity $\times$ perpendicular; (if we suppose the bodies at A) as the velocity, ∴ P. T. in E : P. T. in $e :: \frac{CB}{V.\text{at}A} : \frac{Cb}{V'\text{at}A} :: \frac{CB}{CB} : \frac{Cb}{Cb}$; since the velocities are as the ½ conjugates, *i. e.* in this case as the ½ axes minors.

Hence the periodic times in *all* ellipses round the same centre are equal; for let E and *e* be *any* two ellipses; describe the ellipse ε similar to *e*, and on the same axis major with E, then P. T. in ε = P. T. in *e* because they are similar; and P. T. in ε = P. T. in E because they have the same axis major, ∴ P. T. in *e* = P. T. in E.

Or both cases of the above Corollary may be thus proved. P. T $\doteq \frac{A}{a}$, as $\frac{AC.CB}{\text{velo.} \times \text{per.}} \doteq \frac{AC.CB}{CD \times \frac{AC.CB}{CD}}$ $\doteq$ 1 or is constant. If the absolute forces be different or the bodies move round different centres P. T. $\therefore \frac{A}{a}$, as $\frac{A\,C.\,C\,B}{\text{velo.}\times\text{perp.}}$, as $\frac{A\,C.\,C\,B}{\varphi^{\frac{1}{2}}\times C\,D\times\frac{AC.\,CB}{C\,D}}$, as $\frac{1}{\varphi^{\frac{1}{2}}}$.

NOTES TO SECTION III.

PROPOSITIONS XI. XII. and XIII.

Notes to Props. 11, 12, *and* 13.

142. To make the above Props. general, let $a =$ area described in a given time, then, by Art. 91, we have in general the centripetal force to $= \frac{QR \times a^2}{SP^2 . QT^2}$, *i. e.* in this case $= \frac{a^2}{L \times SP^2}$, which expression is general for bodies moving round different centres of force.

143. If different bodies revolve in conic sections round the *same* centre, then when they are at the same distance from it, the forces will be equal, $\therefore \frac{a^2}{L}$ must be constant, or $a = L^{\frac{1}{2}}$; consequently in this case the force $= \frac{1}{SP^2}$.

144. Let φ represent the absolute force, then accelerating force $= \frac{\varphi}{SP^2} = \frac{a^2}{L \times SP^2}$, $\therefore \varphi = \frac{a^2}{L}$, and $a = L^{\frac{1}{2}} \times \varphi^{\frac{1}{2}}$.

145. To prove the Prop. fluxionally, we have in the ellipse $p^2 = b^2 \times \frac{y}{2a - y}$; in the hyperbola $p^2 = b^2 \times \frac{y}{2a + y}$; and in the parabola $p^2 = ay$; $\therefore$ (1) $\frac{1}{p^2} = \frac{2a}{b^2 y} - \frac{1}{b^2}$; (2) $\frac{1}{p^2} = \frac{2a}{b^2 y} + \frac{1}{b^2}$; (3) $\frac{1}{p^2} = \frac{1}{ay}$, $\therefore F \propto \frac{\dot{p}}{p^3 \dot{y}} \propto$ (in all the three cases) $\frac{1}{L \times y^2}$.

PROPOSITION XIV.

Note to Prop. 14.

146. This Prop. is only applicable to different bodies moving round the *same* common centre. To make it general, let φ = absolute force, then we have $L \propto \frac{QT^2}{QR}$ ultimately, but QR ultimately varies as the accelerating force when the time is given, *i. e.* $\propto \frac{\varphi}{SP^2}$, $\therefore L \propto \frac{SP^2 . QT^2}{\varphi}$, and $SP.^2 QT^2$ or $a^2 \propto L \times \varphi$, and $a \propto L^{\frac{1}{2}} \times \varphi^{\frac{1}{2}}$, which is true for bodies moving round different centres, provided the force $\propto \frac{1}{\text{Dist.}^2}$. The same conclusion was obtained in Art. 144.

Note to Prop. 14.—*Cor.*

147. If bodies revolve in ellipses round different

centres, we have $A \mathrel{\dot{-}} a \times P \mathrel{\dot{-}} L^{\frac{1}{2}} \times \varphi^{\frac{1}{2}} \times P$; *i. e.* in general $A\,C.\ C\,B \mathrel{\dot{-}} L^{\frac{1}{2}}.\ \varphi^{\frac{1}{2}}.\ P$.

PROPOSITION XV.

Note to Prop. 15.

148. This Prop. is only applicable to different bodies moving round the same common centre. To make it general, let φ = absolute force, then $C\,B \mathrel{\dot{-}} L^{\frac{1}{2}} \times A\,C^{\frac{1}{2}}$, $\therefore A\,C.\ C\,B \mathrel{\dot{-}} L^{\frac{1}{2}} \times A\,C^{\frac{3}{2}}$, but (Art. 147) $A\,C.\ C\,B \mathrel{\dot{-}} L^{\frac{1}{2}}.\ \varphi^{\frac{1}{2}}.\ P$, $\therefore L^{\frac{1}{2}}.\ \varphi^{\frac{1}{2}}.\ P \mathrel{\dot{-}} L^{\frac{1}{2}} \times A\,C^{\frac{3}{2}}$, $\therefore P \mathrel{\dot{-}} \frac{A\,C^{\frac{3}{2}}}{\varphi^{\frac{1}{2}}}$, which is true for bodies moving in ellipses round different centres.

PROPOSITION XVI.

Note to Prop. 16.

149. This Prop. is only true for different bodies moving round the same common centre; to make it general we have $V \mathrel{\dot{-}} \frac{S\,P.\ Q\,T}{S\,Y} \mathrel{\dot{-}} \frac{a}{S\,Y} \mathrel{\dot{-}} \frac{L^{\frac{1}{2}} \times \varphi^{\frac{1}{2}}}{S\,Y}$ which is applicable to bodies moving round different centres of force.

Prop. 16.—*Cor.* 4.

150. For $V^2 : v^2 :: \frac{L}{B\,C^2} : \frac{2\,A\,C}{A\,C^2} :: L : \frac{2\,B\,C^2}{A\,C}$

$:: L : L :: 1 : 1$; and $\therefore$ since the velocities in $\odot^s \div \frac{1}{\sqrt{\text{dist.}}}$, the velocities of bodies revolving in ellipses round a common centre will at the mean distance $\div \frac{1}{\sqrt{\text{dist.}}}$

Prop. 16.—*Cor.* 6.

150. For in different points of the same curve $V \div \frac{1}{SY}$; $\therefore$ in parab. $V \div \frac{1}{SY} \div \frac{1}{\sqrt{SP}}$; in the ellipse and hyperbola $V \div \frac{1}{SY} \div \frac{1}{BC\frac{\sqrt{SP}}{\sqrt{PH}}} \div \sqrt{\frac{HP}{SP}}$. Now in the ellipse, as S P or the denominator of this fraction encreases or decreases, H P or the numerator decreases or increases; consequently the fraction $\frac{HP}{SP}$ will vary more than the fraction $\frac{1}{SP}$, and $\therefore$ the velocity will vary in a higher R°. than $\frac{1}{\sqrt{SP}}$; but in the hyperbola, as S P encreases or decreases, H P also encreases or decreases; consequently the fraction $\frac{HP}{SP}$ varies less than the fraction $\frac{1}{SP}$, *i. e.* the velocity varies in a less R°. than $\frac{1}{\sqrt{SP}}$.

Prop. 16.—*Cor.* 7.

152. For in the parabola $V^2 : v^2 :: \frac{4\,SA}{SP.\,SA} : \frac{2\,SP}{SP^2} :: 2 : 1$, $\therefore V : v :: \sqrt{2} : 1$; in the ellipse

$$V^2 : v^2 :: \frac{\frac{2BC^2}{AC}}{BC^2 \cdot \frac{SP}{PH}} : \frac{2SP}{SP^2} :: HP : AC, \therefore V :$$
$v :: \sqrt{HP} : \sqrt{AC} :: \sqrt{2AC - SP} : \sqrt{AC}$
$:: \sqrt{2 - \frac{SP}{AC}} : 1 :: \sqrt{2 -} : 1.$ In hyperbola V :
$v :: \sqrt{HP} : \sqrt{AC} :: \sqrt{2AC + SP} : \sqrt{AC} ::$
$\sqrt{2 + \frac{SP}{AC}} : 1 :: \sqrt{2 +} : 1.$ Hence also velocity in parabola = velocity in ⊙ at ½ the distance. For

$V : v :: \sqrt{2} : 1$
& v : velocity in ⊙r. ½ S P :: $1 : \sqrt{2}$
∴ V = velocity in ⊙ radius ½ S P

But in ellipse $V : v :: \sqrt{2 -} : 1$
& v : v. in ⊙r. ½ S P :: $1 : \sqrt{2}$
∴ V : v. in ⊙r. ½ S P :: $\sqrt{2 -} : \sqrt{2}$
∴ V is less than velocity in ⊙r. ½ S P

And in hyperbola $V : v :: \sqrt{2 +} : 1$
& v : velocity in ⊙r. ½ S P :: $1 \; \sqrt{2}$
∴ V : velocity in ⊙r. ½ S P :: $\sqrt{2 +} : \sqrt{2}$
∴ V is greater than velocity in ⊙r. ½ S P.

Prop. 16.—*Cors.* 8 *and* 9.

153. For let V = velocity in the conic section at the distance S P; v = velocity in a ⊙ at the distance of ½ the *latus rectum*; and v' = velocity in a ⊙ at the distance S P; then since the *latera recta* in the conic section and first ⊙ are equal

V : v :: ½ L : S Y, which is the 8th Corollary, again $v : v' :: \sqrt{SP} : \sqrt{\frac{1}{2}L}$,

$\therefore V : v' :: \sqrt{\frac{1}{2}L \times SP} : SY$, which is the 9th Corollary.

DEDUCTIONS FROM THE PRECEDING PART OF THIS SECTION.

154. *Of the* LINEAR *velocities of bodies revolving in conic sections, the centre of force being in the focus.*

1. *Required a general expression for the velocity of bodies revolving in any of the conic sections.*

(1) In parabola $V^2 \doteq F \times PV \doteq \frac{\varphi}{SP^2} \times 4\,SP \doteq \frac{\varphi}{SP}$, $\therefore V \doteq \frac{\varphi^{\frac{1}{2}}}{SP^{\frac{1}{2}}}$.

(2) In ellipse and hyperbola $V^2 \doteq F \times PV \doteq \frac{\varphi}{SP^2} \times \frac{2\,SP.\,PH}{AC} \doteq \frac{\varphi \times PH}{AC.\,SP}$, $\therefore V \doteq \sqrt{\frac{\varphi \times PH}{AC.\,SP}}$.

Or the same may be deduced from Art. 149, by substituting for L and SY in the proportional Equation $V \doteq \frac{L^{\frac{1}{2}} \times \varphi^{\frac{1}{2}}}{SY}$.

Cor. 1. If different bodies revolve round the same centre, φ is constant; $\therefore$ in parabola $V \doteq \frac{1}{\sqrt{SP}}$; and in ellipse and hyperbola $V \doteq \sqrt{\frac{HP}{AC.\,SP}}$.

Cor. 2. In different points of the same curve we have in parabola $V \propto \frac{1}{\sqrt{SP}}$; and in ellipse and hyperbola $V \propto \sqrt{\frac{HP}{SP}}$.

2. *To compare the velocity in a conic section with the velocity in a circle at the same distance.*

$V \propto \sqrt{PV}$, $\therefore$ velocity in conic section : velocity in $\odot$ at same distance $:: \sqrt{\frac{2\,CD^2}{AC}} : \sqrt{2\,SP} :: \sqrt{\frac{2\,SP.PH}{AC}} : \sqrt{2\,SP} :: \sqrt{HP} : \sqrt{AC}$.

Or the same may be demonstrated as in Prop. 16, Cor. 7.

Cor. 1. Hence velocity in ellipse = velocity in a $\odot$ at the mean distance; for in that case $HP = AC$; the same is also shewn in Prop. 16, Cor. 4.

Cor. 2. Hence also the same conclusions may be deduced as those given in Prop. 16, Cor. 7.

3. *To compare the velocity in any point of the ellipse with the velocity at the mean distance.*

$V \propto \frac{1}{SY}$, $\therefore$ velocity in ellipse : velocity at mean distance $:: \dfrac{1}{CB\sqrt{\frac{SP}{PH}}} : \dfrac{1}{CB} :: \sqrt{HP} : \sqrt{SP}$.

4. *If a body revolve in an ellipse; required the point where the velocity is an arithmetic mean between the greatest and least velocities.*

Let D and d = greatest and least distances, p = perpendicular upon the tangent at the required point; then by the Prob. $\frac{1}{D} + \frac{1}{d} = \frac{2}{p}$, $\therefore p = \frac{2\,Dd}{D+d} =$

$\frac{b^2}{a} = \frac{1}{2}$ L, or at the required point the perpendicular $= \frac{1}{2}$ the *latus rectum;* to find when this is the case we have $p = \frac{b^2}{a} = b \times \sqrt{\frac{x}{2a - x}}$, $\therefore \frac{b^4}{a^2} = \frac{b^2 x}{2a - x}$, $\therefore x = \frac{2ab^2}{a^2 + b^2}$.

5. *If a body revolve in an ellipse; required the point where the velocity is a geometric mean between the greatest and least velocities.*

Here $\frac{1}{D} \times \frac{1}{d} = \frac{1}{p^2}$, $\therefore p^2 = Dd = b^2$ and $p = b$; *i. e.* the required point is at the extremity of the minor axis, or at the mean distance.

6. *Required the point in the parabola, where the decrement of the linear velocity is a maximum.*

By pursuing the method given, (Art. 108) we have $v \doteq \frac{1}{p} \doteq y^{-\frac{1}{2}}$, $\therefore \dot{v} \doteq \frac{\dot{y}}{y^{\frac{3}{2}}}$; but by that Art $\dot{y} = \frac{\sqrt{y^2 - p^2}}{py} = \frac{\sqrt{y^2 - ay}}{a^{\frac{1}{2}} y^{\frac{3}{2}}} = \frac{\sqrt{y - a}}{a^{\frac{1}{2}} y}$, $\therefore \dot{v} \doteq \frac{\sqrt{y - a}}{y^{\frac{5}{2}}}$ which is a maximum by Prob.; $\therefore \frac{y - a}{y^5}$ or $\frac{1}{y^4} - \frac{a}{y^5}$ is a maximum, $\therefore \frac{4\dot{y}}{y^5} - \frac{5a\dot{y}}{y^6} = o$, or $y = \frac{5a}{4}$.

7. *Required the same in the ellipse.*

$v \doteq \frac{1}{p} \doteq \sqrt{\frac{2a - y}{y}} \doteq \sqrt{\frac{2a}{y} - 1}$, $\therefore v \doteq \left(\frac{2a - y}{y}\right)^{-\frac{1}{2}} \times \frac{2a\dot{y}}{y^2} \doteq \frac{2a\dot{y}}{y^{\frac{3}{2}} \sqrt{2a - y}}$; but $\dot{y} \propto \frac{\sqrt{y^2 - p^2}}{py} =$

$$\frac{\sqrt{y^2 - \frac{b^2 y}{2a-y}}}{by\sqrt{\frac{y}{2a-y}}} = \frac{\sqrt{2ay-y^2-b^2}}{by}, \therefore \dot{v} \text{ varies as}$$

$$\frac{\sqrt{2ay-y^2-b^2}}{y^{\frac{5}{2}}\sqrt{2a-y}} = \text{maximum}, \therefore \frac{2ay-y^2-b^2}{y^5 \cdot \overline{2a-y}} \text{ or}$$

$$\frac{1}{y^4} - \frac{b^2}{y^5 \cdot \overline{2a-y}} = \text{maximum}, \therefore \frac{4\dot{y}}{y^5} -$$

$$\frac{10ab^2y^4\dot{y}-6b^2y^5\dot{y}}{y^{10} \times \overline{2a-y}^2} = o; \text{ or } 2y^3 - 8ay^2 + \overline{8a^2 + 3b^2} \cdot y - 5ab^2 = o;$$ from whence x may be found.

8. *Required the point in the parabola, where the paracentric velocity is a maximum.*

By Art. 107, Paracentric velocity $\doteq \frac{\sqrt{y^2-p^2}}{py}$,

$\therefore$ in this case $\doteq \frac{\sqrt{y^2-ay}}{a^{\frac{1}{2}}y^{\frac{3}{2}}} = \text{maximum}, \therefore \frac{y^2-ay}{y^3}$

or $\frac{1}{y} - \frac{a}{y^2} = \text{maximum}, \therefore y = 2a$, *i. e.* the required point is at the extremity of the *latus rectum.*

9. *Required the same in the ellipse and hyperbola.*

Paracentric velocity $\doteq \frac{\sqrt{y^2-p^2}}{py}$, which by Prob. is a maximum, $\therefore \frac{y^2-p^2}{p^2y^2}$ or $\frac{1}{p^2} - \frac{1}{y^2} = \text{maximum}$,

i. e. $\frac{2a \mp y}{b^2y} - \frac{1}{y^2}$, or $\frac{2a}{b^2y} \mp \frac{1}{b^2} - \frac{1}{y^2} = \text{maximum}$, $\therefore y = \frac{b^2}{a} = \frac{1}{2}$ *latus rectum.*

155. *Of the* ANGULAR *velocities of bodies revolving in the conic sections; force tending to the focus.*

1. *Required a general expression for the angular velocity of bodies revolving in any of the conic sections.*

Let a = area described in a given time, then $\angle^r$ velocity $\propto \frac{a}{SP^2} \propto \frac{L^{\frac{1}{2}} \times \varphi^{\frac{1}{2}}}{SP^2}$.

Cor. 1. If different bodies revolve round the same centre, φ is constant, $\therefore$ $\angle^r$ velocity $\propto \frac{L^{\frac{1}{2}}}{SP^2}$.

Cor. 2. In different points of the same curve $\angle^r$ velocity $\propto \frac{1}{SP^2}$.

2. *To compare the* $\angle^r$ *velocity in a conic section with the* $\angle^r$ *velocity in a* ⊙ *at the same distance.*

$\angle^r$ velocity $\propto \frac{L^{\frac{1}{2}}}{SP^2} \propto$ (since the distance is the same) $L^{\frac{1}{2}}$, $\therefore$ $\angle^r$ velocity in the conic section : $\angle^r$ velocity in a ⊙ at the same distance $:: L^{\frac{1}{2}} : \overline{2\,SP}^{\frac{1}{2}}$ $:: \overline{\tfrac{1}{2}L}^{\frac{1}{2}} : \overline{SP}^{\frac{1}{2}}$.

Cor. Hence $\angle^r$ velocity in the conic section = $\angle^r$ velocity in ⊙ at the same distance at the extremity of the *latus rectum.*

3. *To compare the* $\angle^r$ *velocity in any point of the ellipse with the mean* $\angle^r$ *velocity.*

If a circle be described with the focus of the ellipse as centre, and radius = A C or mean distance, the periodic time in this circle will = the periodic time in the ellipse, hence the uniform $\angle^r$ velocity in this ⊙ will represent the mean $\angle^r$ velocity of the body in the ellipse; $\therefore$ since $\angle^r$ velocity $\propto \frac{L^{\frac{1}{2}}}{SP^2}$, we have

$\angle^r$ velocity in any point P : mean $\angle^r$ velocity (or $\angle^r$ velocity in $\odot$ radius = mean distance) ::

$$\frac{\sqrt{\frac{2\,CB^2}{AC}}}{SP^2} : \frac{\sqrt{2\,AC}}{AC^2} :: \frac{1}{SP^2} : \frac{1}{AC.CB}.$$

Cor. Hence the $\angle^r$ velocity in the ellipse = mean $\angle^r$ velocity when $SP^2 = AC.CB$, or when the distance from the focus is a mean proportional between the $\frac{1}{2}$ axes of the orbit.

4. *To compare the $\angle^r$ velocity at the mean distance with the mean $\angle^r$ velocity.*

$\angle^r$ velocity at mean distance : mean $\angle^r$ velocity ::

$$\frac{\sqrt{\frac{2\,CB^2}{AC}}}{AC^2} : \frac{\sqrt{2\,AC}}{AC^2} :: CB : CA.$$

Cor. Hence the $\angle^r$ velocity at the mean distance is less than the mean $\angle^r$ velocity.

5. *The $\angle^r$ velocity round the higher focus of an ellipse of small excentricity is nearly uniform.*

Take Pp (*Fig.* 74) an indefinitely small arc, join PS, pS, and PH, pH; from P draw Pn perpendicular to Sp produced, and Pm perpendicular to Hp; then because the $\angle$ Ppm = $\angle$ SpB = $\angle$ Ppn, and that the $\angle^s$ at n and m are right $\angle^s$ and Pp common, $\therefore$ Pn = Pm; Hence $\angle^r$ velocity round S : $\angle^r$ velocity round H :: $\angle$ PSp : $\angle$ PHp :: $\frac{Pn}{PS} : \frac{Pm}{PH}$ :: $\frac{1}{SP} : \frac{1}{PH} :: \frac{1}{SP^2} : \frac{1}{SP.PH}$; but the $\angle^r$ velocity round S is represented by $\frac{1}{SP^2}$, $\therefore$ the $\angle^r$ velocity round H is represented by $\frac{1}{SP.PH}$ or by $\frac{1}{CD^2}$, which quantity, if the ellipse be of small excentricity, will be very nearly constant.

6. *Required the point in the ellipse where the $\angle^r$ velocity is an arithmetic mean between the greatest and least $\angle^r$ velocities.*

Let D and d = greatest and least distances, x = required distance; then by the Problem $\frac{1}{D^2} + \frac{1}{d^2} = \frac{2}{x^2}$, $\therefore x^2 = \frac{2 D^2 d^2}{D^2 + d^2} = \frac{2 b^4}{D^2 + d^2}$. But $D^2 + d^2 + 2 D d = 4 a^2$, $\therefore D^2 + d^2 = 4 a^2 - 2 D d = 4 a^2 - 2 b^2$, $\therefore x^2 = \frac{2 b^4}{4 a^2 - 2 b^2} = \frac{b^4}{2 a^2 - b^2}$, and $x = \frac{b^2}{\sqrt{2 a^2 - b^2}}$.

7. *Required the point in the ellipse where the $\angle^r$ velocity is a geometric mean between the greatest and the least.*

Here $\frac{1}{D^2} \times \frac{1}{d^2} = \frac{1}{x^4}$, $\therefore x^2 = D d$, and $x = \sqrt{D d} = b$.

8. *Required the point in the parabola where the decrement of the $\angle^r$ velocity is a maximum.*

By Art. 109, the decrement of the $\angle^r$ velocity = $\frac{\sqrt{y^2 - p^2}}{p y^4}$, which by Problem is a maximum, $\therefore \frac{y^2 - p^2}{p^2 y^8}$, or $\frac{1}{p^2 y^6} - \frac{1}{y^8}$, or $\frac{1}{a y^7} - \frac{1}{y^8}$ is a maximum, $\therefore y = \frac{8 a}{7}$.

9. *Required the same in the ellipse.*

Let $y = SP$, $v = PH$; then as before $\frac{1}{p^2 y^6} - \frac{1}{y^8}$ is a maximum, or $\frac{v}{b^2 y^7} - \frac{1}{y^8}$ is a maximum, $\therefore$ $\frac{b^2 y^7 \dot{v} - 7 b^2 v y^6 \dot{y}}{b^4 y^{14}} + \frac{8\dot{y}}{y^9} = o$, or $\frac{y \dot{v} - 7 v}{b^2} + \frac{8\dot{y}}{y} = o$; but $v = 2a - y$, and $\dot{v} = -\dot{y}$, $\therefore$ by substitution, $\frac{y + 7.\overline{2a - y}}{b^2} - \frac{8}{y} = o$, $\therefore 3y^2 - 7ay + 4b^2 = o$; from which equation y may be found.

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156. *Of Centripetal and Centrifugal Forces in the Conic Sections, the centre of force being in the focus.*

1. *Required a general expression for the centrifugal force in the conic sections.*

Let a = area described in a given time, then centrifugal force $\doteq \frac{a^2}{SP^3} \doteq \frac{L \times \varphi}{SP^3}$.

Cor. 1. If different bodies revolve round the same centre, φ is constant, $\therefore$ the centrifugal force $\doteq \frac{L}{SP^3}$.

Cor. 2. In different points of the same curve, centrifugal force $\doteq \frac{1}{SP^3}$.

2. *To compare centripetal and centrifugal forces in the conic sections.*

(1) In parabola; centripetal force : centrifugal :: $2SP^3 : SY^2 \times PV :: SP : 2SA :: SP : \frac{1}{2}L$.

(2) In ellipse and hyperbola; centripetal force : centrifugal :: $2\,SP^3 : SY^2 . PV :: 2\,SP^3 : BC^2$. $\frac{SP}{PH} \times \frac{2\,SP.PH}{AC} :: SP : \frac{BC^2}{AC} :: SP : \frac{1}{2}L$.

Cor. Hence centripetal force = centrifugal at the extremity of the *latus rectum*.

3. *Force in any conic section : force in circle at the same distance, and moving with the same* $\angle^r$ *velocity* :: SP : $\frac{1}{2}$ L.

For by Art. 142, force $\propto \frac{a^2}{L \times SP^2} \propto$ (since the $\angle^r$ velocity and distance, and consequently a are the same in both cases) $\frac{1}{L}$, $\therefore$ force in conic section : force in $\odot$ at same distance, and moving with the same $\angle^r$ velocity $:: \frac{1}{L} : \frac{1}{2\,SP} :: SP : \frac{1}{2}L$.

Or the same may be deduced from the last Example; for the force in the $\odot$ at the same distance, and moving with the same $\angle^r$ velocity, is equal to the centrifugal force in the curve, but it has been shewn that centripetal force : centrifugal :: SP : $\frac{1}{2}$ L, $\therefore$ &c.

MISCELLANEOUS PROBLEMS TO THE TWO LAST SECTIONS.

1. *Required the Ratio of the quantities of matter in planets which have secondaries revolving round them.*

Let φ = absolute force = quantity of matter in primary; D = $\frac{1}{2}$ axis of the ellipse described by the secondary, or = mean distance of the secondary from the primary, P = periodic time of the secondary; then by Art. 148, $P^2 \propto \frac{D^3}{\varphi}$ $\therefore \varphi \propto \frac{D^3}{P^2}$; φ may $\therefore$ be assumed $= \frac{D^3}{P^2}$; from whence we shall get the quantity of matter of the several planets in proportional N°s.

2. *Required the Ratio of the densities of planets which have secondaries revolving round them.*

Let d = density of the primary, r = radius of primary, s = sin. of the ∠ under which r appears at the distance D to radius unity; then since density $\propto \frac{\text{quan. M}^r}{\text{magni.}}$, we have $d \propto \frac{\varphi}{r^3} \propto \frac{D^3}{P^2 . r^3}$; but $\frac{r^3}{D^3} = s^3$, $\therefore d \propto \frac{1}{P^2 . s^3}$; assume $\therefore$ d $= \frac{1}{P^2 . s^3}$, and we shall get the density of the planets in proportional N°.

3. *Required the Ratio of the weights of equal bodies on the surfaces of planets having secondaries revolving round them.*

The weight of any body $\propto$ quantity of matter × accelerating force; $\therefore$ since the bodies are equal by supposition, the weight will be as the force with which the planets attract it, *i. e.* weight $\stackrel{.}{-} \frac{\varphi}{r^2} \stackrel{.}{-} \frac{D^3}{P^2 . r^2}$, or $\stackrel{.}{-} d \times r$. This will also give the R°. of the spaces fallen through in 1″ at the surface of the planets; for space $\stackrel{.}{-}$ accelerating force, when the time is given.

Note.—The density, &c. of planets, which have not satellites revolving round them, can only be found by observing the effects which those planets produce upon the other planets in disturbing their motion.

4. ***How must the force be changed in an ellipse, to make a body move in a parabola.***

$F \stackrel{.}{-} \frac{V^2}{PV} \stackrel{.}{-}$ (in this case where V is given) $\frac{1}{PV}$;

$\therefore$ F in ellipse : F in parabola $::$ 4 SP : $\frac{2 SP . PH}{AC}$ $::$ 2 AC : PH.

5. ***If an n^{th} part of the earth were taken away, what change would be produced in the moon's orbit, and in what R°. would her periodic time be encreased, the moon's orbit before the change being supposed circular.***

Since $F \stackrel{.}{-} \frac{1}{D^2}$, the new orbit will be one of the conic sections, the centre of the earth being in the focus. Let $\therefore$ APQ *(Fig. 75)* be the original, and ARM the new orbit, and let the change take place when the body is at A; then since the original orbit is a ⊙, the point A will be an apse of the conic section ARM. Now $F \stackrel{.}{-} \frac{V^2}{PV} \stackrel{.}{-}$ (in this case) $\frac{1}{PV}$;

$\therefore$ force before change or (1) : force after $(1 - \frac{1}{n})$ $::$

$\frac{2\,CD^2}{AC} : 2\,SA$, *i. e.* $n : n-1 :: \frac{2\,CD^2}{AC} : 2\,SA :: \frac{2\,SA.\,SM}{AC} : 2\,SA :: SM : AC :: 2\,AC - AS : AC$; $\therefore AC = \frac{n-1}{n-2}.\,SA$. Now (1) let $n = 2$, *i. e.* let $\frac{1}{2}$ the earth be taken away, then will A C be infinite, or the curve in that case will be a parabola. (2) Let n be less than 2, *i. e.* let more than $\frac{1}{2}$ the earth be taken away, then will A C be finite but negative; ∴ curve is an hyperbola. (3) Let n be greater than 2, or less than $\frac{1}{2}$ the earth be taken away; then will A C be finite and positive, or the curve in that case will be an ellipse, whose $\frac{1}{2}$ axis major $= \frac{n-1}{n-2}.\,SA$. To find the change in the periodic time; we have $P \propto \frac{\overline{\frac{1}{2}\text{ ax. maj.}}^{\frac{3}{2}}}{\varphi^{\frac{1}{2}}}$, ∴ P. T before change : P. T. after $:: \frac{SA^{\frac{3}{2}}}{1} : \frac{\overline{\frac{n-1}{n-2}.\,SA}^{\frac{3}{2}}}{\overline{\frac{n-1}{n}}^{\frac{1}{2}}} :: \overline{n-2}^{\frac{3}{2}} : n^{\frac{1}{2}}.\,\overline{n-1}$; which R°. is only real and finite when n is greater than 2, or when the curve is an ellipse.

Cor. In the two last cases $\overline{\frac{1}{2}\text{ ax. min.}}^2 = AS.\,SM = AS.\,\overline{AM - AS} = \frac{n}{n-2}.\,\overline{SA}^2$.

6. *Supposing the velocity with which a body would revolve in a circle at the earth's surface to be given; what must be the velocity, the direction continuing the same, that the excentricity of the orbit may be* 1000 *miles.*

Let A P Q (former figure) be a great ⊙ of the earth, A R M the ellipse described by the body, S the centre of the earth or focus of the ellipse, S C the excentricity; put A S $= r$, S C $= a$, V $=$ velocity in A R M at A, $v =$ velocity in A P Q at A; then since $F \propto \frac{V^2}{PV}$, and that F is the same at A in both cases, $V^2 \propto PV$; hence $V^2 : v^2 :: \frac{2\,CD^2}{AC} : 2\,SA ::$ $\frac{2\,SA.\,SM}{AC} : 2\,SA :: SM : AC :: 2a + r : a + r$,

$$\therefore V = v\sqrt{\frac{2a + r}{a + r}}.$$

Cor. If a be infinitely greater than r, or the path of the body be a parabola $V = v\sqrt{2}$.

7. *Centrifugal force at the equator, arising from the earth's rotation round its axis: the centrifugal force in any parallel of latitude* :: $\overline{rad.}^2$: $\overline{cos.\ latitude}^2$, *supposing the earth a perfect sphere.*

Let P p (*Fig.* 76) be the earth's axis, Æ Q the equator, A B any parallel of latitude, and take Q D and B n proportional to the centrifugal forces at Q and B; revolve B n into B m and $m\,n$, then will B m represent that part of the centrifugal force at B which diminishes the force of gravity; then since $F \propto \frac{R}{P^2}$ $\propto$ (since P is here given) R, we have

Q D : B n :: C Q : A B :: rad. : cos. latitude
And B n : B m :: C B : A B :: rad. : cos. latitude

$\therefore$ Q D : B m :: r^2 : $\overline{\text{cos. latitude}}^2$.

Cor. 1. Hence, since Q D and r^2 are constant, the diminution of gravity, or that part of the centrifugal force which diminishes gravity, in going from pole to equator, $\propto$ $\overline{\text{cos. latitude}}^2$.

Cor. 2. In what latitude does centrifugal force $= \frac{1}{m}$th the centrifugal force at the equator. Here $r^2 : \overline{\text{cos. latitude}}^2 :: m : 1$, $\therefore$ cos. latitude $= \frac{r}{\sqrt{m}}$.

8. *Required the velocity of the earth round its axis, that the centrifugal force in lat.* 60° *may* = *force of gravity there.*

Let V = required velocity, C = centripetal force or gravity, c = centrifugal force at equator, c' = centrifugal force in latitude 60°; then since $F \propto \frac{V^2}{R}$ we have

$$C : c :: 2\,m\,r : V^2$$

$$\text{but } c : c' :: \overline{\text{rad.}}^2 : \overline{\text{cos. latitude}}^2 :: 4 : 1$$

$\therefore C : c' :: 8\,m\,r : V^2$; but C = c' by hypothesis, $\therefore V = \sqrt{8\,m\,r}$.

9. *Required to find how the weight of the same body varies on different parts of the earth's surface.*

Let P = time of the earth's rotation round its axis; p = periodic time of a body revolving at the earth's surface; ϑ = cos. latitude; C = centripetal force or force of gravity; c = centrifugal force at the equator; c' = centrifugal force in any other parallel of latitude; then

$$C : c :: \frac{1}{p^2} : \frac{1}{P^2} :: P^2 : p^2$$

$$\& \; c : c' :: \overline{\text{rad.}}^2 : \vartheta^2$$

$$\therefore C : c' :: \overline{\text{rad.}}^2\, P^2 : p^2 . \vartheta^2$$

& C : C $- c'$ (or comparative weight) $:: \overline{\text{rad.}}^2\, P^2 : \overline{\text{rad.}}^2\, P^2 - p^2 . \vartheta^2$; but the 1st and 3d terms are constant, $\therefore$ weight $\propto \overline{\text{rad.}}^2\, P^2 - p^2 . \vartheta^2$.

Or thus. Let r = radius of the earth, then since $F \propto V^2$ when R is given

$$C : c :: V^2 : v^2 :: 2mr : v^2$$

$$\& \; c : c' :: \overline{\text{rad.}^2} : \vartheta^2$$

$$\therefore C : c' :: \overline{\text{rad.}^2}\, 2mr : \vartheta^2 v^2$$

$$\& \; C : C - c' \text{ (or compar. weight)} :: \overline{\text{rad.}^2}\, 2mr : \overline{\text{rad.}^2}\, 2mr - \vartheta^2 v^2$$

$$\therefore \text{weight} \propto \overline{\text{rad.}^2}\, 2mr - \vartheta^2 v^2.$$

Cor. To compare the force of gravity in any two latitudes.

Let C = cos. of latitude in one of the places, c = D°. at the other, then since force of gravity by last Cor. $\propto \overline{\text{rad.}^2}\, 2mr - \vartheta^2 v^2$, where ϑ = cos. latitude, gravity at one place : gravity at the other :: $\overline{\text{rad.}^2}\, 2mr - C^2 v^2 : \overline{\text{rad.}^2}\, 2mr - c^2 v^2$, or :: $\overline{\text{rad.}^2}\, P^2 - p^2 C^2 : \overline{\text{rad.}^2}\, P^2 - p^2 c^2$.

10. *Required the Ratio of the times of oscillation of a pendulum in any two given latitudes, supposing the earth a sphere.*

Let C and c be the cosines of the two latitudes, T and t the times of oscillation of the pendulum at those latitudes, P and p as in the last Prob., then since time of oscillation $\propto \frac{1}{\sqrt{\text{Fce.}}}$, when the length of the pendulum is given, we have by Cor. Prob. 9. $T : t :: \sqrt{\overline{\text{rad.}^2}\, P^2 - p^2 c^2} : \sqrt{\overline{\text{rad.}^2}\, P^2 - p^2 C^2}$.

Cor. If the two places be the pole and the equator, we have c = cos. 0 = rad., and C = cos. 90°. = 0; $\therefore T : t :: \sqrt{P^2 - p^2} : P$.

11. *In a given latitude a pendulum will oscillate once in a second, supposing the earth not to revolve round its axis:—Required the ∠r motion round its axis that the pendulum may oscillate once in two seconds.*

Let v = required velocity round its axis; c = cos.

latitude; F = force of gravity at 1st, or when the earth is at rest; f = force of gravity when it revolves round its axis; then since time of oscillation $\doteq \frac{1}{\sqrt{F}}$, when the length of the pendulum is given, $1 : 2 :: \frac{1}{\sqrt{F}} : \frac{1}{\sqrt{f}} :: \sqrt{f} : \sqrt{F} :: \sqrt{\overline{\text{rad.}}^2\, 2\,m\,r - c^2\,v^2}$ $: \sqrt{\overline{\text{rad.}}^2\, 2\,m\,r}$ (by Cor. Prob. 9); $\therefore \overline{\text{rad.}}^2\, m\,r = 4\,\overline{\text{rad.}}^2\, m\,r - 2\,c^2\,v^2$, $\therefore v^2 = \frac{3\,\overline{\text{rad.}}^2\, m\,r}{2\,c^2}$, and $v = \frac{\text{rad.}}{c} \times \sqrt{\frac{3\,m\,r}{2}}$.

12. *Supposing a pendulum in latitude 60° to oscillate seconds, when the earth revolves round its axis with a velocity of* v *feet per second; required the velocity of the earth round its axis, that the pendulum may oscillate once in two seconds.*

Let V = required velocity, then, as before, comparative gravity $\doteq \overline{\text{rad.}}^2\; 2\,m\,r - 9^2\,v^2 \doteq$ (in this case where Latitude = 60°.) $8\,m\,r - v^2$, $\therefore 1 : 2 :: \frac{1}{\sqrt{F}} : \frac{1}{\sqrt{f}} :: \sqrt{f} : \sqrt{F} :: \sqrt{8\,m\,r - V^2} : \sqrt{8\,m\,r - v^2}$; $\therefore 8\,m\,r - v^2 = 32\,m\,r - 4\,V^2$, and $V = \sqrt{\frac{24\,mr + v^2}{4}}$.

13. *If a body is set rolling from* B (Fig. 77) *down the quadrant* B P D, *with the velocity acquired in falling through the given space* A B; *to determine the point where it will leave the quadrant, and the point where it will meet the horizontal plane.*

When the body leaves the quadrant it will describe a parabola, let it leave the circle in P; then P is a point both in the parabola and circle, and P B D is a circle of curvature to the parabola at P, since P V $\doteq$

$\frac{V^2}{F}$; Hence velocity at P = velocity acquired in falling down $\frac{1}{4}$th of the chord of curvature or $\frac{1}{2}$ P F; but it also = velocity down A B + B E; ∴ A B + B E $= \frac{PF}{2} = \frac{BC - BE}{2}$; ∴ B E $= \frac{BC - 2AB}{3}$ = vers. sin. of arc described.

Again, from A draw A N parallel to the horizon, which line is the directrix of the parabola P p; make ∠ S P x = ∠ N P x, and P S = P N, and S is the focus; with S as centre and A C as radius, describe a circle cutting the horizontal line C p in p; p is the point required. For S p = C A = $p\,o$; ∴ p is a point in the parabola.

Cor. If A B = $\frac{1}{2}$ B C, B E = o, or the body will fly off in a tangent at B; if A B be greater than $\frac{1}{2}$ B C, then B E is negative, *i. e.* ver. sin. is negative, or the Prob. is impossible.

14. *Suppose a body to begin to move from the point* C (Fig. 78) *of the cycloid* A C P; *to find the point where the body will leave the curve.*

Let P be the point required; then as before (since P F = $\frac{1}{4}$ chord of curvature of cycloid, and ∴ of parabola since P V = $\frac{V^2}{F}$); P F or E D = B E, *i. e.* A D − A E = A E − A B, ∴ A E = $\frac{AD + AB}{2}$.

15. *A body whirled round by a string* C A (Fig. 79) *in a vertical plane just keeps the string extended at* A; *required the proportion of the tension of the string at* B *to the weight of the body.*

By the Prob. the centrifugal force at A is just = the weight of the body, and ∴ the velocity at A is = that acquired in falling through D A = $\frac{AC}{2}$; also the velocity at B = that acquired through DA + AB or $\frac{5\,AC}{2}$; ∴ since centrifugal force ∝ V^2, when r is given, *i. e.* ∝ space fallen through; centrifugal force at B : centrifugal force at A, or weight of the body, :: 5 : 1; but the tension of the string at B is made up of the centrifugal force at B together with the weight of the body; ∴ tension of string at B : weight of body, :: 6 : 1.

16. *If a body suspended by a string oscillate through a quadrant (the extremity of the quadrant being the lowest point); to compare the tension of the string with the weight of the body in any point of the descent.*

Let P (*Fig.* 80) be any point of the descent, W = whole weight of the body, w = that part of it which is employed in stretching the string, C = centrifugal force of the body at P, and x = sin. ∠ PAB to radius r. Then

$$W : w :: PE : PD :: r : x$$

$$C : W :: \frac{V^2}{r} : 2m :: \frac{4mx}{r} : 2m :: 2x : r$$

$$\therefore C : w :: 2 : 1$$

$$\& \; C + w \text{ or tension at P} : w :: 3 : 1$$

$$\text{but } w : W :: x : r$$

$$\therefore \text{tension at P} : W :: 3x : r.$$

Or thus. Let gravity or the weight of the body be represented by $2m$; then $w = \frac{2mx}{r}$; also centrifugal force upon the same scale $= \frac{V^2}{r} = \frac{4mx}{r}$; ∴ $C + w$ = tension at P $= \frac{6mx}{r}$; ∴ tens. : weight $:: \frac{6mx}{r} : 2m :: 3x : r$.

Cor. Hence the tension of the string at the lowest point = three times the weight of the body.

17. *Required the same in the cycloid.*

Let gravity or the weight of the body be represented by $2m$, and put DG (*Fig.* 81) = a, and DF = x; then $2m : w :: DG : DE :: \overline{DG}^{\frac{1}{2}} : \overline{DF}^{\frac{1}{2}} :: a^{\frac{1}{2}} : x^{\frac{1}{2}} \therefore w = \frac{2m x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$; also C upon the same scale $= \frac{V^2}{\frac{1}{2}PV} = \frac{4mx}{2x} = 2m$; $\therefore$ C + w or tension at P $= \frac{2m x^{\frac{1}{2}}}{a^{\frac{1}{2}}} + 2m = 2m \times \frac{a^{\frac{1}{2}} + x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$; $\therefore$ tension at P : weight $:: 2m. \frac{a^{\frac{1}{2}} + x^{\frac{1}{2}}}{a^{\frac{1}{2}}} : 2m :: a^{\frac{1}{2}} + x^{\frac{1}{2}} : a^{\frac{1}{2}}$.

Cor. At the lowest point, tension : weight :: 2 : 1.

18. *Let* AP (Fig. 82) *be a slender rod in the form of a curve, whose axis* NA *is perpendicular to the horizon, and let a ring be put upon it at any point* P; *suppose the rod to revolve about* AN *with such a velocity that the ring may remain at rest at* P; *required the nature of the curve* AP, *that the ring may also remain at rest at every other point of the rod.*

Draw PT a tangent to the curve at P, put NP = y, TN = subtangent = t, V = velocity of the rod at P; then if gravity be represented by $2m$, we have centrifugal force at P $= \frac{V^2}{y}$ = suppose to P D; $\therefore \frac{V^2}{y}$: that part of the force which urges the body up the rod, or PE :: PT : y, $\therefore$ PE $= \frac{V^2}{PT}$; again, gravity, or $2m$ = PC : that part of it which urges the body down the rod, or P B :: P T : t; ∴ $PB = \frac{2mt}{PT}$; but since the body remains at rest, P E $= PB$, *i. e.* $\frac{V^2}{PT} = \frac{2mt}{PT}$, and $V = \sqrt{2m \times TN}$; in like manner if p be any other point, the velocity necessary to make the ring rest at $p = \sqrt{2m \times tn}$; ∴ in order that the body may remain at rest both at P and p, velocity at P must be to velocity at p :: $\sqrt{TN} : \sqrt{tn}$; but velocity at P : velocity at p :: P N : pn; ∴ in order that the body may remain at rest both at P and p, T N must be to tn :: $PN^2 : pn^2$, or the subtangent must be as the square of the ordinate, *i. e.* the curve must be a parabola.

Cor. Hence if a vessel of water revolve about its axis, the cavity formed in the fluid by the revolution of the vessel will be a paraboloid; for every particle of the water forming the surface of the cavity remains at rest by the supposition, and ∴ by the foregoing Prob. must lie in the surface of the paraboloid.

19. *The curve* A B P *being a parabola, and the rest as before; let it be required to find the proper velocity with which any point* P *must revolve, that the ring placed at* P *may remain at rest.*

Let x = space fallen through by gravity to acquire the required velocity; then as before we have $PE = \frac{V^2}{PT} = \frac{4mx}{PT}$, and $PB = \frac{2m \times TN}{PT}$ $= \frac{2m \times 2AN}{PT}$, ∴ $\frac{4mx}{PT} = \frac{4m \times AN}{PT}$, ∴ $x =$ A N, or the body must fall through a space equal to the abscissa of the curve.

Cor. If A P be any other curve, $x = \frac{TN}{2}$; or the space fallen through must = ½ the subtangent.

20. *A cylindrical vessel is filled with water; with what velocity must it be whirled round its axis that $\frac{1}{2}$ the water may be thrown out.*

By Cor. Prob. 18, when the cylinder is turned round, the surface of the water in the vessel is a paraboloid; and since the cylinder is full at first, the quantity of water thrown out will always be equal to the content of the paraboloid thus formed: now the greater the velocity of the cylinder, the greater will be the quantity of water thrown out; *i. e.* the lower will the vertex descend; and since by the Prob. just half the water is thrown out, the cylinder must be whirled with such a velocity that the vertex of the paraboloid may descend till it just touch the bottom of the cylinder; for in that case the quantity of water thrown out = the content of the paraboloid inscribed in the cylinder = $\frac{1}{2}$ content of the cylinder. Let F A M (*Fig.* 83) be the surface of the water; then since after it has assumed this position it is supposed to remain at rest, any particle as P is at rest. Let x = space fallen through to acquire the velocity of rotation at P; then by proceding as in last Prob., x = A N; and for the same reason the velocity of a particle at M, or the velocity of the cylinder = velocity acquired down G A or the height of the vessel.

21. *A cylindrical vessel of a given magnitude is filled with water; with what velocity must it be whirled round its axis, that the water may just cover $\frac{1}{2}$ the base.*

Let A B C D (*Fig.* 84) be the cylinder, A m n B the cavity formed in the water, let the paraboloid A m L B be completed, and put H L = x, H G = h, then $AB^2 : mn^2 :: LH : LG :: x : x - h$; but by Prob. $AB^2 : mn^2 :: 2 : 1$; $\therefore 2 : 1 :: x : x - h$, and $2 : 2 - 1\ (1) :: x : h$, $\therefore x = 2h$; hence by proceeding as in the last Prob. we shall have the velocity of a particle remaining at rest at B, or the velocity of the cylinder = that acquired in falling down H L or $2h$.

22. *A frustum of a cone of given dimensions, and having its smaller end downwards, is filled with water; with what velocity must it revolve round its axis, that all the water may be expelled.*

Let A M N B (*Fig.* 84) be the frustum; then in order that all the water may fly out, the velocity of the vessel must be such that the fluid would, if permitted, form itself into the paraboloid A B N L M circumscribing the frustum; put A B = a, M N = b, L H = x, and H O = h; then $a^2 : b^2 :: x : x - h$, and $a^2 : a^2 - b^2 :: x : h$, $\therefore x = \frac{a^2 h}{a^2 - b^2}$ = space fallen through to acquire the velocity sought.

FINIS.

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ERRATA.

Page 32, line 4, for *by Lem.* 12, read *by Conics*.
63, 15, after *and* insert *hence*.
65, Lem. 4, for DEF read $A\,a\,E$, and for $d\,e\,f$ read $P\,p\,T$.
98, line 4, for $C\,c$ read $c\,C$, also in line 8.
99, 28, for $C\,G$ read $G\,C$.
102, 10, for *varies* read *vary*.
113, 5, for $2\,R$ read $Q\,R$.
Id. 20, for $SP^2 : SY^2$ read $SY^2 : SP^2$.
121, 4, for $-Fy$ read $-F\dot{y}$.
124, ult. for y read $\dot{y}$.
125, 31, for *preserving* read *persevering*.
131, 24, for $x\,x\,\dot{y}$ read $x\,\dot{x}\,\dot{y}$.
133, 10, for $\frac{-x^2 \times b^2}{y^3}$ read $\frac{-\dot{x}^2 \times b^2}{y^3}$
148, 19, for v read $\dot{v}$.

Fig. 1

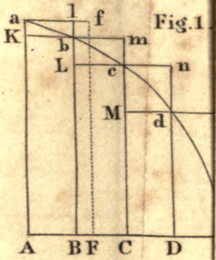
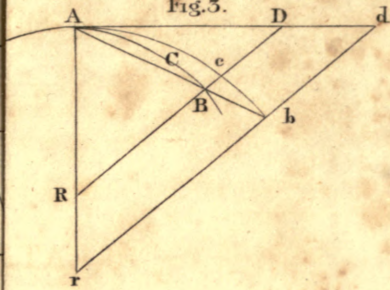
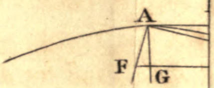


Fig. 3.





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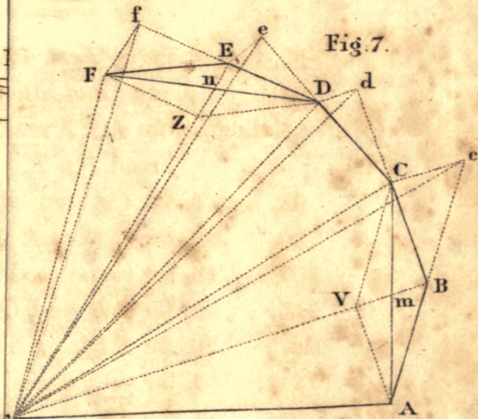


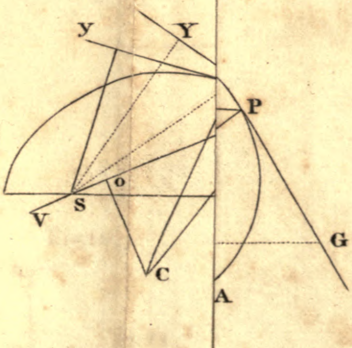
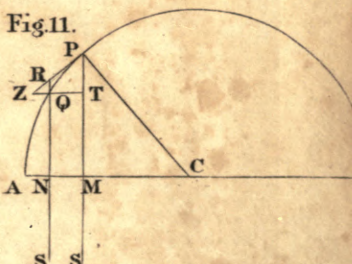


Fig. 11.



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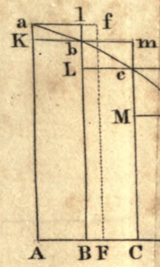

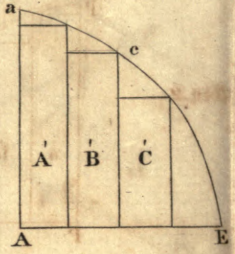

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Fig.4.

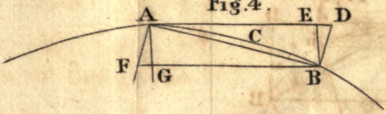

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Fig.8.

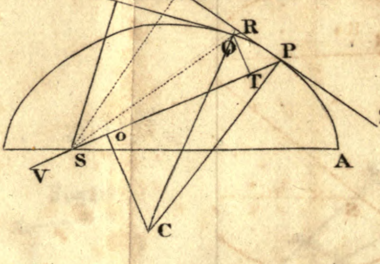
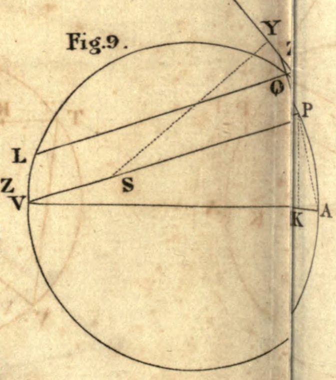


Fig.9.



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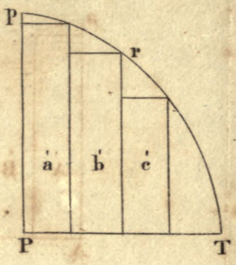
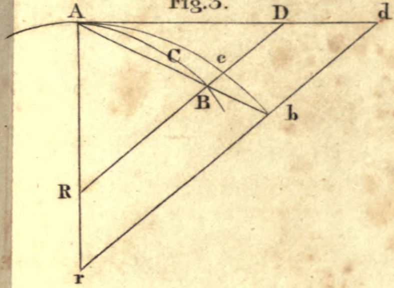
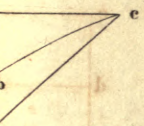


Fig. 3.





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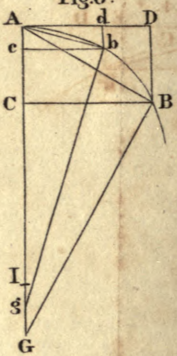
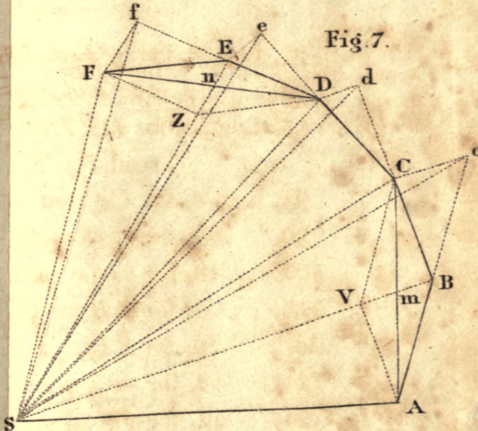
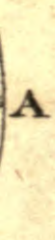


Fig. 7.





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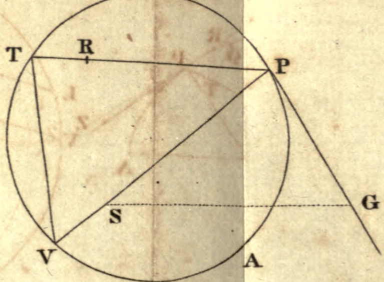
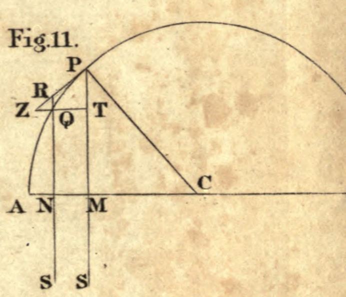


Fig. 11.



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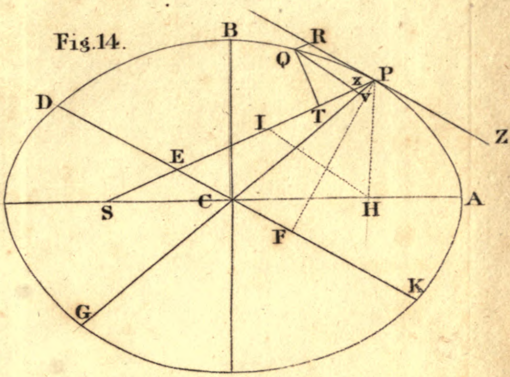

Fig.17.

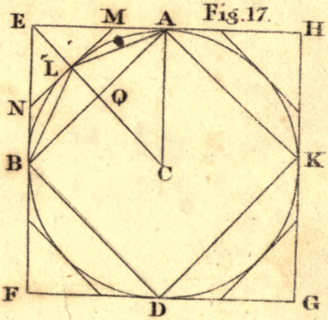

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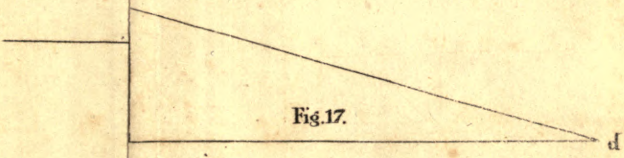
Fig.17.



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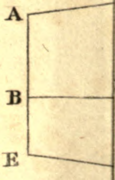

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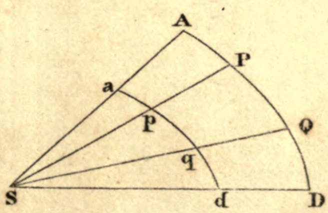

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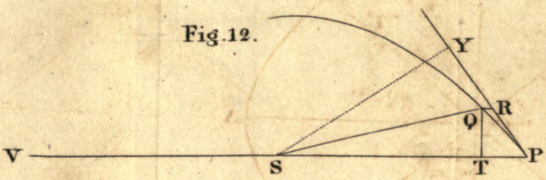

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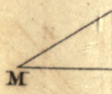

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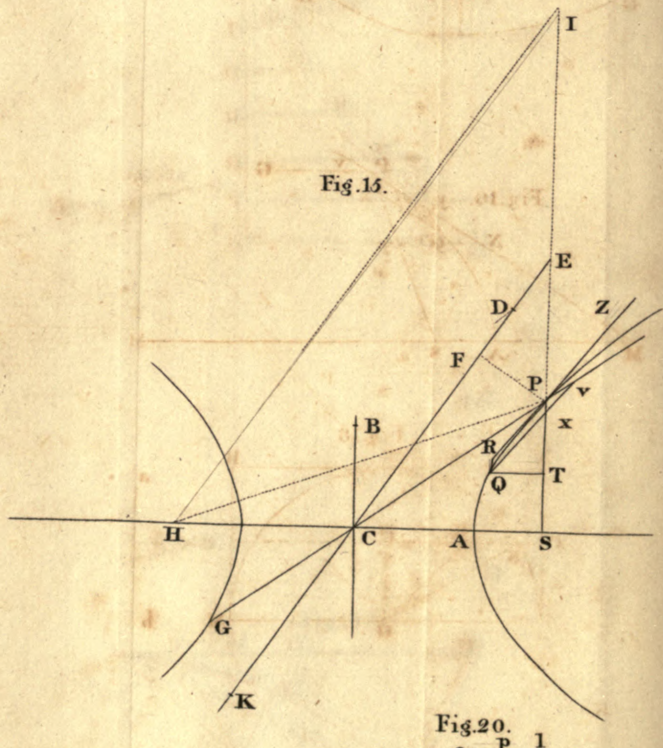

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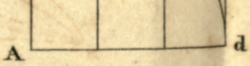

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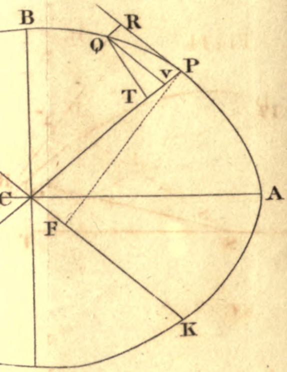

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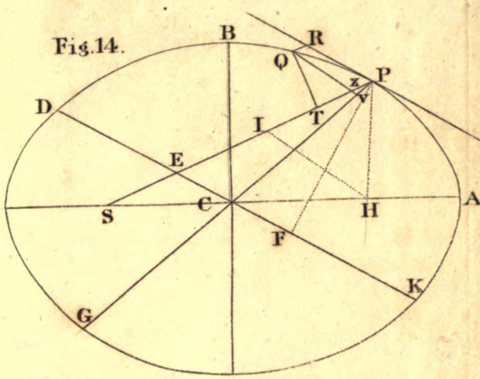

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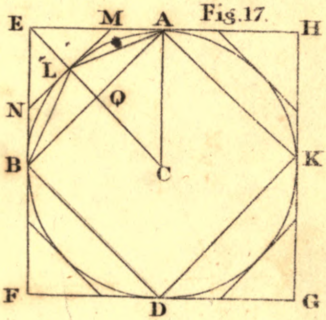

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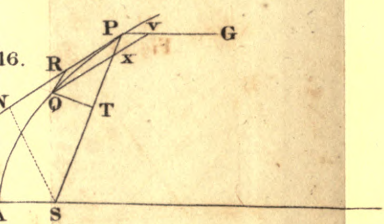
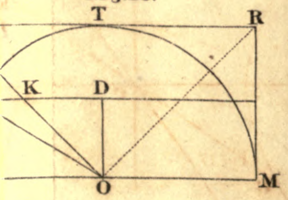
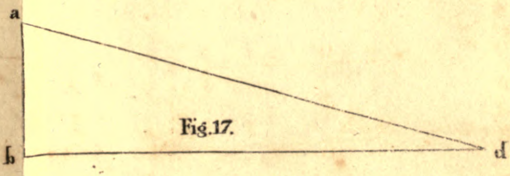


Fig.18.





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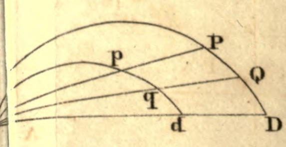

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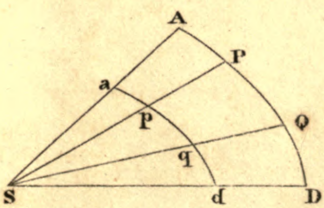

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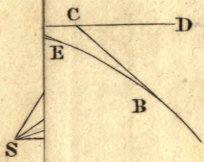

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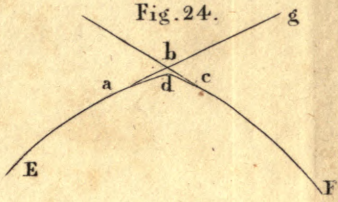
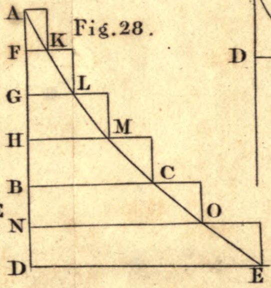


Fig.2

Fig.28.



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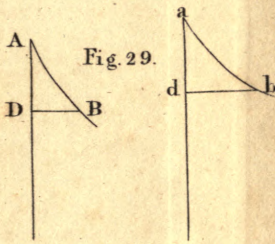
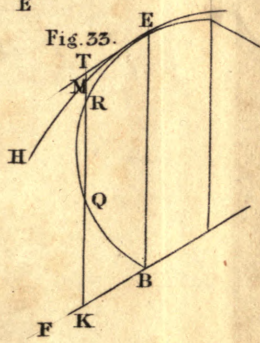


ig.32.

Fig.33.



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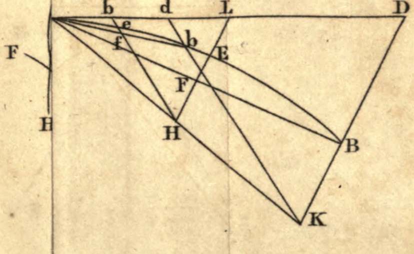


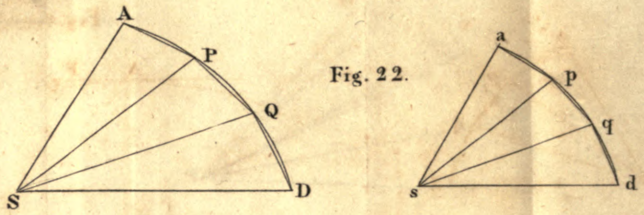
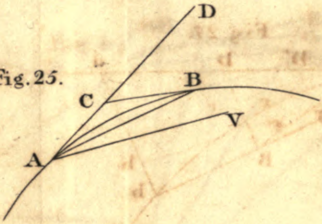


Fig. 22.

Fig. 25.



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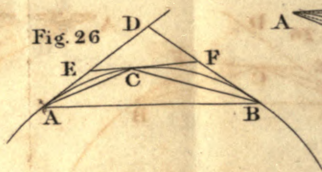

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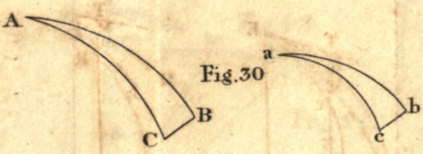
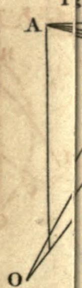


Fig. 31.

Fig. 34.

Fig. 35.



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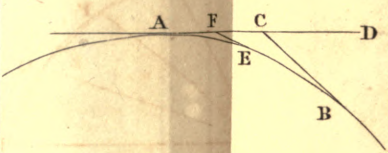

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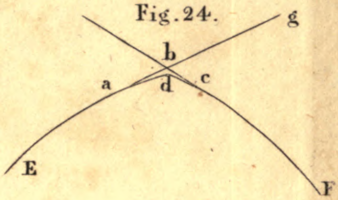

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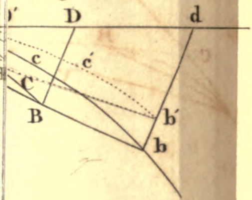

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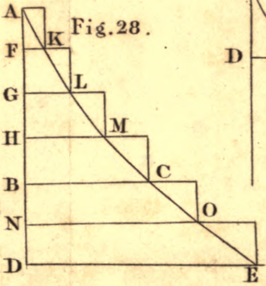

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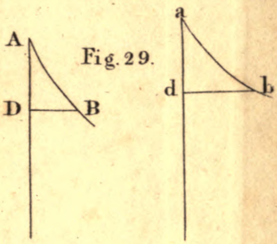

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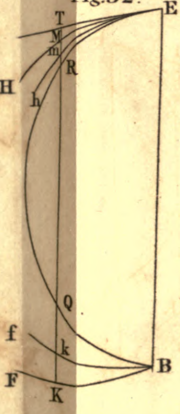

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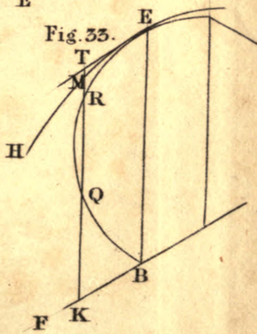

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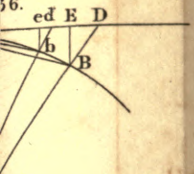

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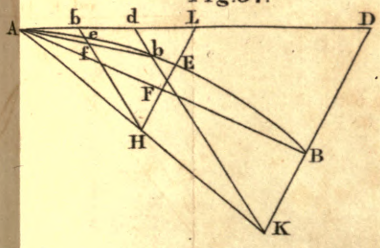

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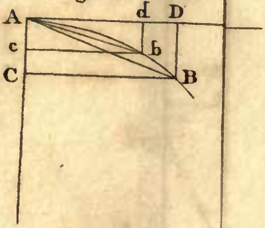
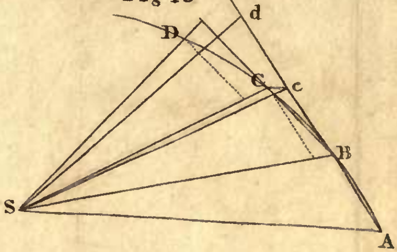


Fig 43



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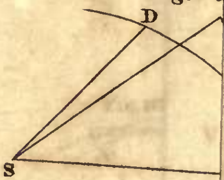

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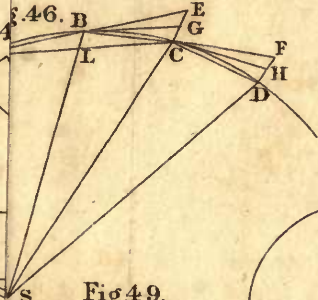

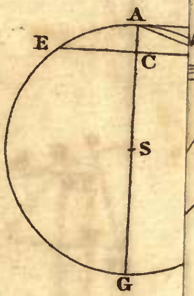

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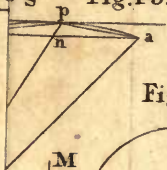

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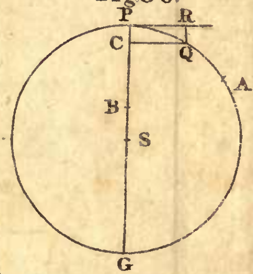

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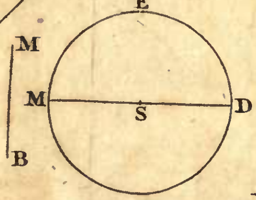

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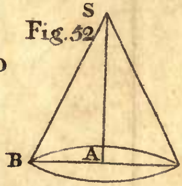

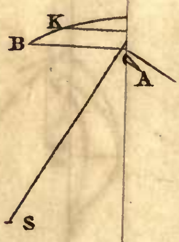

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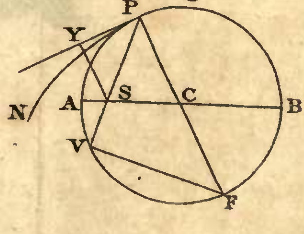

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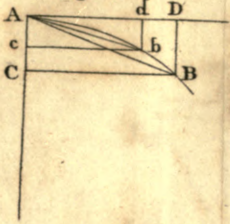
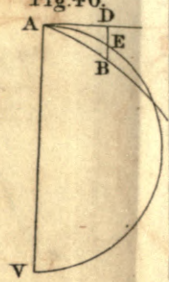


Fig.39.

Fig.40.



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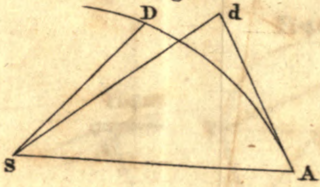

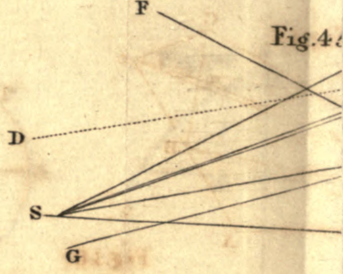

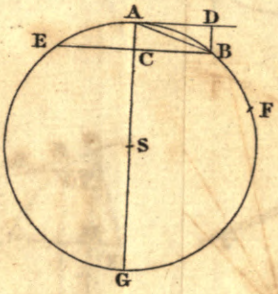

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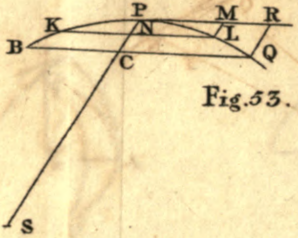

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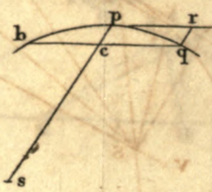
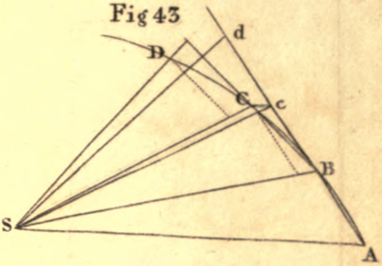
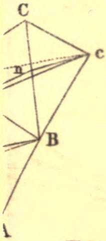


Fig. 41.

Fig. 42.

Fig. 43





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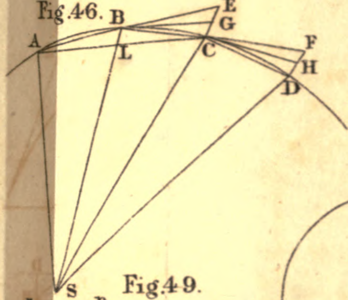

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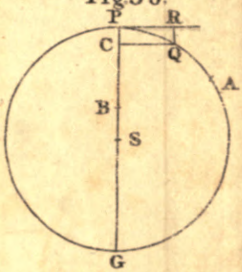

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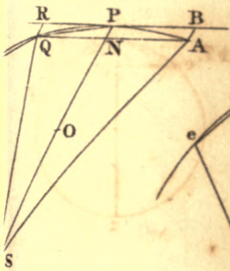

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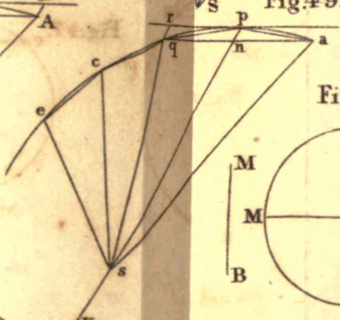

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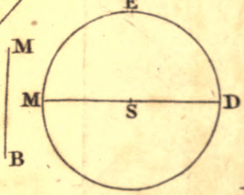
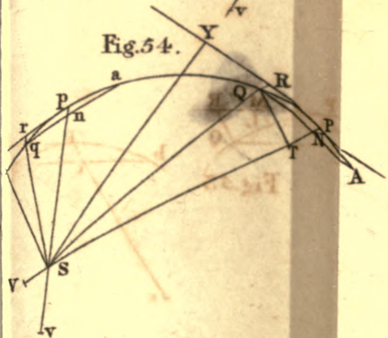


Fig. 52

Fig. 54.



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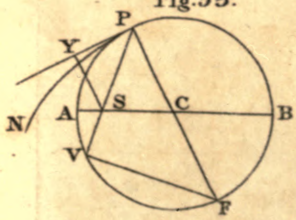

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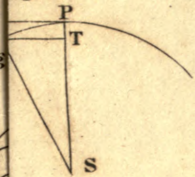
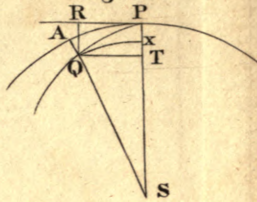


Fig

Fig. 60.



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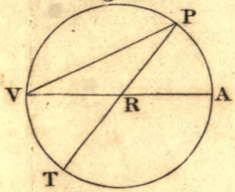

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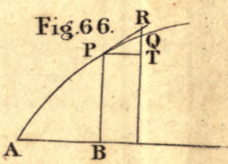
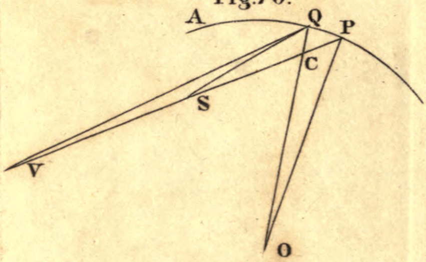
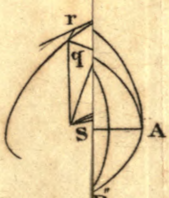


Fig. 61

Fig. 70.





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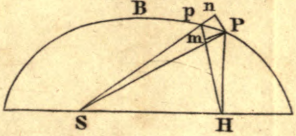

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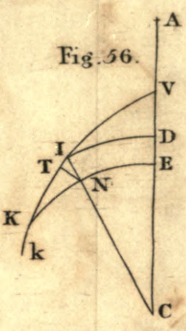

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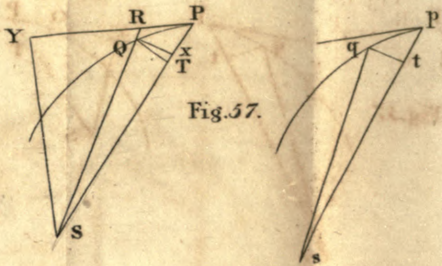

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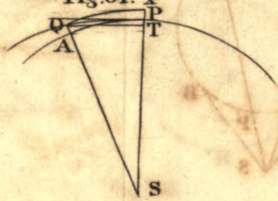

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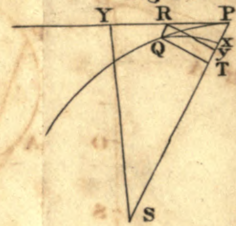

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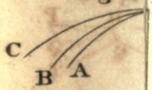

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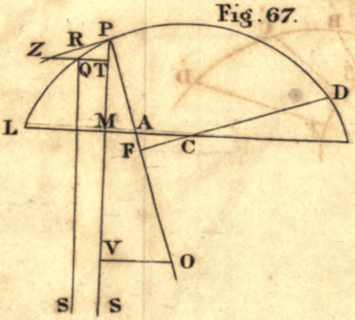

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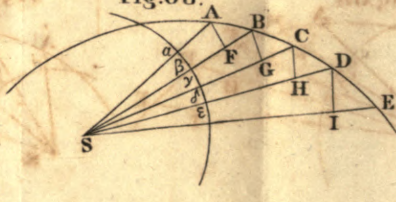

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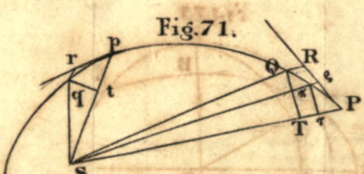

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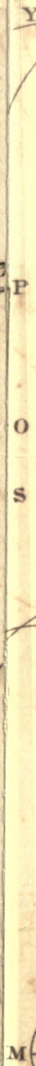

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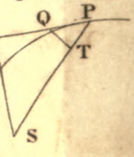

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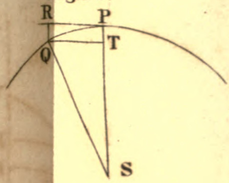

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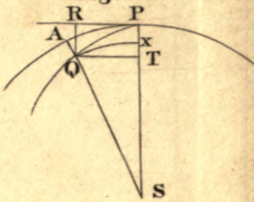

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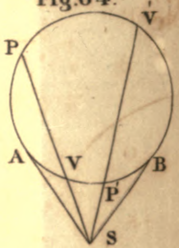

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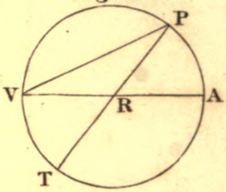

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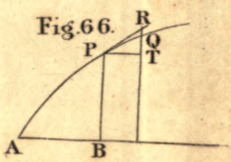

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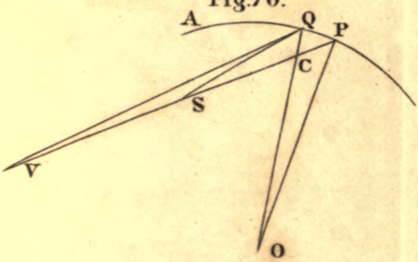
Fig.70.



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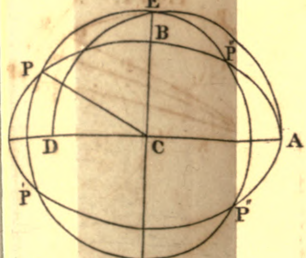

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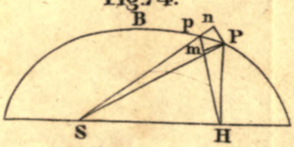

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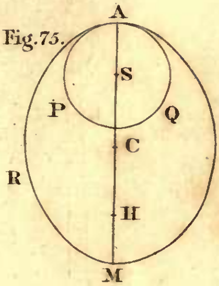


Fig.76.

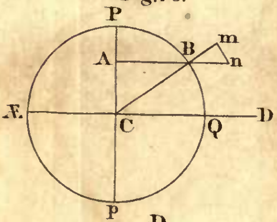


Fig.77.

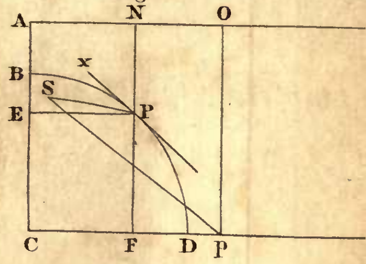


Fig.78.

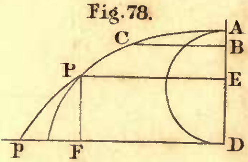


Fig.79.

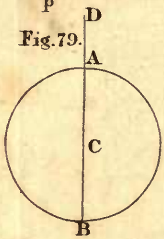


Fig.80.

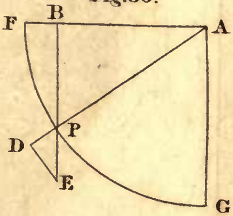


Fig.81.

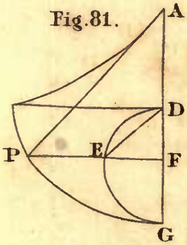


Fig.82.

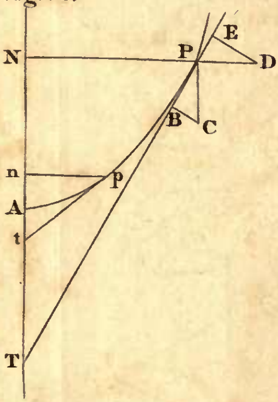


Fig.83.

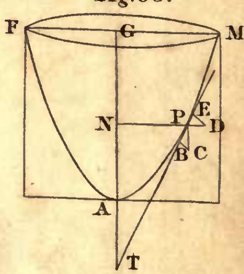
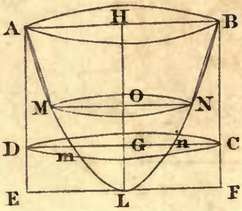


Fig.84.



Proof of eq. [illegible] p. 117.